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Constrained control for systems on matrix Lie groups with uncertainties

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Abstract

In this paper, the constrained control of systems evolving on matrix Lie groups with uncertainties is considered. The proposed methodology is composed of a nominal Model Predictive Control (MPC), and a feedback controller. The previous work on the control of systems on manifolds is applied to design the nominal MPC, which generates the nominal trajectory. In the nominal MPC, the state and input constraints on the Lie group are transformed into the constraints on the Euclidean space. While to deal with uncertainties, the feedback control used to track the nominal trajectory is designed directly on the Lie group. The tracking error in the feedback control is proved to be bounded in invariant sets, which are further used to revise the constraints in nominal MPC. We prove that the input-to-state stability of the entire system under the proposed control methodology with respect to the disturbances can be achieved. The proposed methodology is applied to the constrained attitude control of rigid bodies with uncertainties. In the application example, the detailed mathematical proof and the comparative numerical simulation are presented to demonstrate the feasibility of the proposed methodology.

KEYWORDS

attitude control, matrix Lie group, robust control, tube-based model predictive control

1 | INTRODUCTION

1.1 | Motivation and background

Many systems are subject to constraints, including state and input constraints, and uncertainties. The state and input constraints are critical. The uncertainties may let the system violate the state and input constraints, deteriorating the safety of the system. How to address the state and input constraints under uncertainties is, therefore, a meaningful and challenging problem.

The tube-based MPC is a useful tool that can deal with the state and input constraints of a dynamic system with disturbances.[1,2] Mayne et al. proposed the idea of tube-based MPC.[1] As indicated in References 1, 3, in the tube-based MPC scheme, the MPC is actually built upon the virtual nominal dynamics of the system. The output of the nominal MPC does not directly apply on the real dynamics, but through a feedback control. Under such scheme, the stability of the nominal system under constraints is easier to guarantee. By carefully designing the feedback control, the convergence of the overall system can also be obtained. The tube-based MPC has aroused great interests in the past decades. Farina et al. investigated the tube-based robust sampled-data MPC for linear continuous-time systems.[3] Nonlinear MPC for tracking constant and dynamic reference signals based on tube-based MPC are also investigated.[4-6] Mario E. Villanueva et al. proposed min-max differential inequality to describe the support function of positive robust forward invariant tube in tube-based MPC.[7] V. Raković et al. investigated the safe polyhedral tubes constructed via simple algebraic operations.[8] Cannon et al. proposed a stochastic tube-based MPC for linear systems.[9] Trodden et al. proposed a switching tube-based MPC to guarantee safe and stable operation of disturbed switching linear systems.[10]

Because of the progress in tube-based MPC, it has been applied to a variety of disturbed dynamic systems.[11] Dimarogonas et al. investigated the decentralized control of uncertain nonlinear multi-agent systems using tube-based MPC.[12,13] They also studied the constrained control problem of underwater vehicles by tube-based MPC.[14] Chen et al. addressed a trajectory-tracking control problem for mobile robots by combining tube-based MPC.[15] Sakhdari et al. and Gao et al. applied the tube-based MPC in autonomous vehicles to enhance the safety of the vehicle under uncertainties.[16,17] Kobilarov et al. proposed a tube-based MPC whose tube is expressed by ellipsoids.[18] Yue et al. proposed a robust tube-based model predictive control for lane change maneuver of tractor-trailer vehicles.[19] Lu et al. proposed a robust self-triggered MPC scheme for linear systems based on the tube-MPC.[20] Some researchers also applied tube-based MPC to the control problem of networks, for example, Reference 21.

On the other side, the state space of many systems is non-Euclidean space. Due to the topological difference between the non-Euclidean manifold and Euclidean space, control of systems with non-Euclidean configuration space is still a challenging problem. The control methodology can mainly be categorized into the two groups: the coordinate-based control and the geometric control.[22,23] The former method usually replies on local coordinates of the Lie group. Taking $SO(3)$ as an example, the local coordinates include Euler angles,[24] quaternions,[25] exponential coordinates,[23,26] and so forth. Because of the topological obstacles, the global continuous control on non-Euclidean Lie groups usually does not exist.[27-29] To investigate the global control problem of systems on Lie groups, various methodologies have been proposed, for example, the hybrid system theories are adopted to construct global but non-continuous tracking error which is further used to design the global controller.[30,31] Besides, by constructing the continuous tracking error, the geometric control developed directly on the non-Euclidean Lie group can achieve almost global stability.[32-35]

Recent attempts to control systems on manifolds include the method by embedding the manifold into ambient Euclidean space,[36,37] where the design procedure is usually divided into two steps. First, the given manifold is embedded into an ambient Euclidean space and the system dynamics is stably extended on the Euclidean space. Then the controller is designed on the ambient Euclidean space. As the system dynamics on the manifold is stably extended, the stability of the controlled system on the manifold can therefore be obtained. Such a methodology does not need a local coordinate chart on the manifold, so it can avoid frequent problems induced by local coordinate charts. In the authors' previous work, the MPC on manifold via embedding is also considered.[38] By stably extending the system dynamics from manifold to ambient Euclidean space, the MPC techniques on Euclidean space can be applied directly. However, the previous work does not consider the uncertainties of the system, which may make the actual trajectory differs from the nominal trajectory. In this way, the constraints of the systems may be violated. The constrained control problem of the system on the manifold with uncertainties is therefore a meaningful problem. It is noted that there are some significant challenges to address this problem for systems in non-Euclidean space. In order to guarantee the safety of the system, one may need to express the tube, that is, the invariant set of the tracking error. However, the tube of the tracking error is not preserved anymore after the extension of the dynamics from manifold to ambient Euclidean space. Therefore, it is difficult to express the tube if we apply the control approach for systems on the manifold via embedding.

1.2 | Contributions

In this paper, we aim to solve the constrained control problem for the systems evolving on the matrix Lie group with uncertainties. We will extend the previous methodology which embeds the matrix Lie group into ambient Euclidean space. Inspired by the methodology of tube-based MPC, the entire control framework is composed of a nominal MPC and a feedback controller. We will design the nominal trajectory in the nominal MPC by extending the systems on the Euclidean space, and the feedback controller directly on the Lie group. Considering the disturbance, the tube of the tracking error on Euclidean space will also be defined. By transferring the tube from the Euclidean space to the Lie group, we will show that in such a framework, the constraints of system can be ensured to be fulfilled. The overall methodology is summarized as in Figure 1.

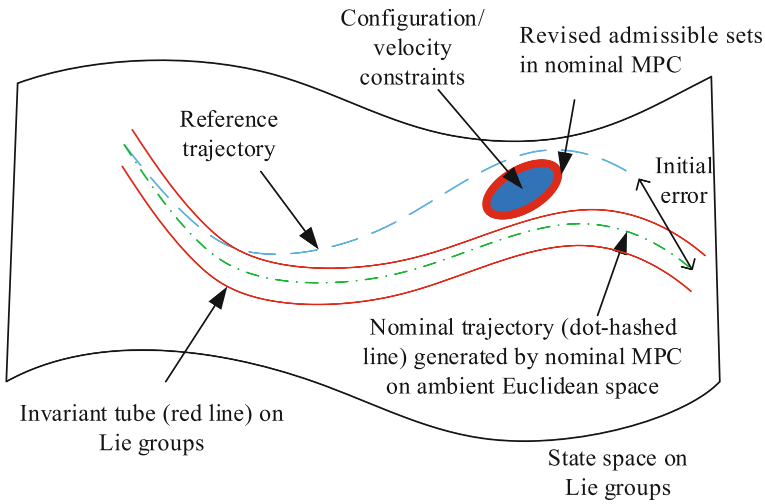

FIGURE 1 The proposed methodology of this paper. The nominal trajectory (dot-hashed line) is generated by nominal MPC on Euclidean space. The actual trajectory falls in the invariant tube (red line). The actual feedback controller is designed on Lie groups. The nominal trajectory converges to the reference trajectory asymptotically with the nominal MPC. While the actual trajectory always keeps in the invariant tube along the nominal trajectory

A preliminary version was published in our previous work.[39] The conference paper[39] introduces the basic idea of this paper. However, it does not include the formal results. In this work, we formally define the problem by introducing series of definitions and assumptions which were not included in previous work.[39] We thoroughly analyze the input-to-sate stability of the system which was also not introduced in previous work.[39] The input-to-state stability of systems on Lie groups under the proposed control is the key part of the formal results. Different from Reference 39, the theoretical completeness of this paper is guaranteed. Besides, the literature review and controller design are also incremented in this paper compared to previous work.[39] The comparison simulation which highlights the advantage of the proposed method is also presented in this paper for the first time.

In summary, the contribution of this paper can be outlined as follows:

1. We propose a framework to address the constrained control problem for systems on matrix Lie groups with uncertainties. The proposed methodology does not rely on any local coordinate set of the Lie group and can apply the existing MPC technique on Euclidean spaces. The theoretical completeness of the proposed framework is guaranteed in the problem definition.
2. The stability of the closed loop nominal systems on Lie groups is proved. Furthermore, the input-to-state stability of systems on Lie groups under the proposed control approach with respect to disturbances is strictly obtained.
3. The proposed methodology is applied to the constrained attitude control of rigid bodies with uncertainties. In the application example, the feasibility and advantage of the proposed methodology is demonstrated via comparative simulation.

This paper is organized into five sections. Section 2 presents the background and the problem definition. In Section 3, the framework of the tube-based MPC on the manifold is designed and analyzed. In Section 4, the proposed methodology is applied to the constrained attitude control of rigid bodies. Conclusions are drawn in Section 5.

1.3 | Notation

Given a matrix Lie group G and sets $S_1, S_2 \subset G$, we define the following set operations

$$S_1 \odot S_2 = \{s_1 s_2 : s_1 \in S_1, s_2 \in S_2\}$$
$$S_1 \oslash S_2 = \{s_1 : s_2 s_1 \in S_1, \forall s_2 \in S_2\}$$

Also we define the following set operations for sets in Euclidean spaces,

$$S_1 \oplus S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$$
$$S_1 \ominus S_2 = \{s_1 : s_2 + s_1 \in S_1, \forall s_2 \in S_2\} \tag{1}$$

The group identity of the matrix Lie group G is denoted by I, and G is supposed to be embedded into a Euclidean space $\mathbb{R}^{n\times n}$. The Lie algebra of G is denoted by $\mathfrak{g}$. The vector space $\mathbb{R}^{n\times n}$ is split into two orthogonal subspace $\mathfrak{g}$ and $\mathfrak{g}^{\perp}$ such that $\mathbb{R}^{n\times n} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$ where $\mathfrak{g}^{\perp}$ is the orthogonal component of $\mathfrak{g}$ in Euclidean space $\mathbb{R}^{n\times n}$. We can define the orthogonal projection maps from $\mathbb{R}^{n\times n}$ to $\mathfrak{g}$ and $\mathfrak{g}^{\perp}$ as,

$$\mathbb{R}^{n\times n} \ni v \mapsto v^{\perp} \in \mathfrak{g}^{\perp}, \mathbb{R}^{n\times n} \ni v \mapsto v^{\parallel} \in \mathfrak{g}. \tag{2}$$

In general, for a variable $*$, the suffix $*_0$ is used to describe the reference signal, the notation $\overline{*}$ is used to denote the nominal signal, while the notation with tilde $\widetilde{*}$ is used to denote the error $* - \overline{*}$.

Furthermore, let us define $\mathrm{Ad}_A B = ABA^{-1}$ for all $A \in \mathrm{GL}(n)$, $B \in \mathbb{R}^{n\times n}$. Then, $\mathrm{Ad}_g\ \xi \in \mathfrak{g}$ for all $g \in G \subset \mathrm{GL}(n)$ and $\xi \in \mathfrak{g} \subset \mathbb{R}^{n\times n}$, where G is a subgroup of $\mathrm{GL}(n)$ and $\mathfrak{g}$ is the Lie algebra of G. The Euclidean inner product on $\mathbb{R}^{n\times n}$ is defined by $\langle A, B\rangle = trace(A^T B)$ for all $A, B \in \mathbb{R}^{n\times n}$. The norm of a matrix on $\mathbb{R}^{n\times n}$ is given by $\|x\| = \sqrt{\langle x, x\rangle}, x \in \mathbb{R}^{n\times n}$.

A continuous function $f_k : [0, a) \mapsto [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $f_k(0) = 0$. A continuous function $f_{kl} : [0, a) \times [0, 0) \mapsto [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the mapping $f_{kl}(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $f_{kl}(r, s)$ is decreasing with respect to s and $f_{kl}(r, s) \to 0$ as $s \to 0$.

For a dynamic system

$$\dot{x} = f(x, u)$$

where $x \in \mathcal{X}$ is the state and $u \in \mathcal{U}$ is the input. The system is called ISS (input-to-state stable) if there exist $f_{kl} \in \mathcal{KL}$ and $f_k \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$[40]

$$\|x(t)\| \le f_{kl}(\|x(t_0)\|, t - t_0) + f_k\left(\sup_{t_0 \le \tau \le t} \|u(\tau)\|\right)$$

for $\forall t \ge t_0$.

2 | BACKGROUND AND PROBLEM FORMULATION

2.1 | System dynamics and preliminaries

Systems evolving on an m-dimensional matrix Lie group G can be expressed by the following equation of motion (EOM),

$$\dot{g} = g\xi$$
$$\dot{\xi} = f(\xi, u) + d, \tag{3}$$

where $g \in G, \xi \in \mathfrak{g}$, and $u \in \mathbb{R}^m, f(\cdot, \cdot)$ is a vector field, $d \in \mathfrak{g}$ is the bounded disturbance induced by modeling uncertainties, and external disturbances. We assume d satisfies $\|d\| \le b_1$.

Remark 1. The second component in (3) is the dynamic equation. The disturbance is added in the dynamic equation. The kinematic equation is given by the left invariant vector field on Lie group, and is regarded as a coordinate transformation between the velocity and derivative of configuration. Although for disturbed systems, the measured configuration and velocity may differ from the actual configuration and velocity, in the estimator it can let the estimated velocity and the time derivative of estimated configuration satisfy the *relationship* reflected by the kinematic equation. Thus we do not need to add disturbance on kinematic equation. Using such expression, the disturbance is actually counted in the dynamic equation. Including the disturbance only on the inner loop is widely adopted in the literature.[41-44]

Denote the reference trajectory of the system by

$$\mathbb{R} \ni t \mapsto (g_0(t), \xi_0(t)) \in G \times \mathfrak{g} \tag{4}$$

and the corresponding reference input of the system by

$$\mathbb{R} \ni t \mapsto u_0(t) \in \mathbb{R}^m. \tag{5}$$

The reference state and input satisfy the system dynamics without disturbance,

$$\begin{aligned} \dot{g}_0 &= g_0\xi_0 \\ \dot{\xi}_0 &= f(\xi_0, u_0). \end{aligned} \tag{6}$$

Assumption 1. There exist constants $\beta_{gmax} \geq \beta_{gmin} > 0$ such that any $g \in G$ satisfies $\beta_{gmin} I \preccurlyeq gg^T \preccurlyeq \beta_{gmax} I$.

Assumption 2. The reference trajectory g_0 is smooth. The reference input u_0 can be solved from g_0 and $\dot{g}_0$.

Assumption 2 is satisfied for systems such as the deferentially flat systems.[45]

Assumption 3. There exists a C^2 function

$$\mathbb{R}^{n\times n} \ni x \mapsto V(x) \geq 0 \in \mathbb{R}$$

with the following properties:

(a) $V^{-1}(0) = G$,
(b) $V(gx) = V(x)$ for all $x \in \mathbb{R}^{n\times n}, g \in G$,
(c) $\nabla^2 V(I)$ is positive definite in the transversal direction, that is, $\nabla^2 V(I) \cdot (y, y) > 0$ for all $y \in \mathfrak{g}^\perp \setminus 0$.

Remark 2. It is often the case that G is expressed as a level set $G = F^{-1}(c_0)$ of a function $F : \mathbb{R}^{n\times n} \to \mathbb{R}^\ell$ for some $c_0 \in \mathbb{R}^\ell$, where F is also G-invariant, that is, $F(gx) = F(x)$ for all $g \in G$ and $x \in \mathbb{R}^{n\times n}$. In this case, we can choose V as $V(x) = k\|F(x) - c_0\|^2$ with $k > 0$. For example, for rotation matrices we can use $F(R) = R^T R$ for $R \in \{A \in \mathbb{R}^{3\times 3} | \det A > 0\}$ so that $SO(3) = F^{-1}(I)$ and $V(R) = k\|R^T R - I\|^2$.

Under Assumption 3, we extend the system dynamics (3) to the ambient Euclidean space $\mathbb{R}^{n\times n}$ by embedding the matrix Lie group G into the Euclidean space $\mathbb{R}^{n\times n}$ as

$$\begin{aligned} \dot{x} &= x\xi - \alpha \nabla V(x) \\ \dot{\xi} &= f(\xi, u) + d, \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^{n\times n}$, $\alpha > 0$.

The tracking error trajectory can be defined on the Euclidean space accordingly

$$\mathbb{R} \ni t \mapsto (E(t), \Xi(t)) := (xg_0^{-1} - I, \xi - \xi_0) \in \mathbb{R}^{n\times n} \times \mathfrak{g}. \tag{8}$$

If we let the trajectory of the system (3) track the reference trajectory, the tracking error dynamics of (3) can therefore be expressed as

$$\begin{aligned} \dot{E} &= (g_0 + Eg_0)\Xi g_0^{-1} \\ \dot{\Xi} &= f(\Xi + \xi_0, u) - f(\xi_0, u_0) + d. \end{aligned} \tag{9}$$

It is noticed that the system (9) also evolves on the Lie group, not on the Euclidean space. By applying the technique of embedding the matrix Lie group G into Euclidean space $\mathbb{R}^{n\times n}$, we can obtain the following equation evolving on $\mathbb{R}^{n\times n}\times\mathbb{R}^m$,

$$\begin{aligned}\dot{E} &= (g_0 + Eg_0)\Xi g_0^{-1} - \alpha\nabla V(g_0 + Eg_0)g_0^{-1}\\ \dot{\Xi} &= f(\Xi + \xi_0, u) - f(\xi_0, u_0) + d.\end{aligned} \tag{10}$$

In this way, we say that the tracking error dynamics of (3) is embedded into the Euclidean space stably.

2.2 | Problem formulation and overall control architecture

In this paper we will consider the control problem of dynamic systems evolving on matrix Lie groups, under state constraints, input boundedness, and uncertainties. The control problem can therefore be expressed as follows.

Problem 1. Consider the system evolving on matrix Lie groups governed by the EOM (3). Given specific configuration constraint $g\in\mathcal{X}$, and specific velocity constraint $\xi\in\mathcal{V}$, input constraint $u\in\mathcal{U}$, for reference state and input $(g_0,\xi_0)\in\mathcal{X}\times\mathcal{V}\subset G\times\mathfrak{g}$, deign control input $u:\mathbb{R}\ni t\mapsto u(t)\in\mathbb{R}^m$ which guarantees the tracking error ISS at origin with respect to disturbance d, while fulfilling all the above constraints for all disturbance satisfying (3).

The configuration error E can further be divided into the parallel direction error $E^{\parallel}$ and the transversal direction error $E^{\perp}$. Given the reference trajectory $g_0(t)$ satisfying $\alpha_1 I \preccurlyeq g_0(t)g_0(t)^T \preccurlyeq \alpha_2 I$ for all t, we linearize (10) along the reference trajectory, the tracking error dynamics can be expressed as,

$$\begin{aligned}\dot{E}^{\perp} &= -\alpha((\nabla^2 V(I)\cdot E^{\perp})(g_0 g_0^{\perp})^{-1})^{\perp}\\ \dot{E}^{\parallel} &= g_0\Xi g_0^{-1} - \alpha\,(\nabla^2 V(I)\cdot E^{\perp})(g_0 g_0^{T})^{-1}\big)^{\parallel}\\ \dot{\Xi} &= \frac{\partial f}{\partial \xi}(\xi_0,u_0)\Xi + \frac{\partial f}{\partial u}\delta_u + d,\end{aligned} \tag{11}$$

where $\delta_u = u - u_0$.

As stated in Reference 36, the first equation in (11) is exponentially stable at the origin. It is also possible to design control based on the linearized system (11). However, in order to solve Problem 1, we need to carefully consider the set of tracking errors, which may influence the admissible input and state set. As it is difficult to estimate the boundedness of the tracking error for the linearized system, we will therefore develop a methodology which generates the nominal trajectory based on (11), and tracks the nominal trajectory based on (3) directly. More specially, a virtual dynamics which is the nominal dynamics by excluding the disturbance from (11) will be used to design the nominal MPC on the ambient Euclidean space. As the generated nominal trajectory always lies in the Lie groups, the actual dynamics (3) is then used to design the actual controller on Lie groups. In this way, the invariant tube representing the tracking error between the actual and nominal trajectory can be obtained to revise the admissible state and input set in the nominal MPC. The overall architecture of the proposed control methodology is depicted in Figure 2.

In the proposed control scheme, the feasibility and stability of the nominal closed loop system can be achieved without considering the uncertainties, as the nominal system is independent from the uncertainties. However, the actual system depends on the uncertainties d. By showing how the actual system converges to the nominal trajectory under uncertainties, the ISS of the overall system can be proved accordingly.

Assumption 4. There exist constants $\beta_{amax}\geq\beta_{amin}>0$ and $\beta_{bmax}\geq\beta_{bmin}>0$ such that the following inequality holds for the linearized dynamics along the nominal trajectory

$$\beta_{amin}\leq\left\|\frac{\partial f}{\partial\xi}\right\|\leq\beta_{amax},\beta_{bmin}\leq\left\|\frac{\partial f}{\partial u}\right\|\leq\beta_{bmax}. \tag{12}$$

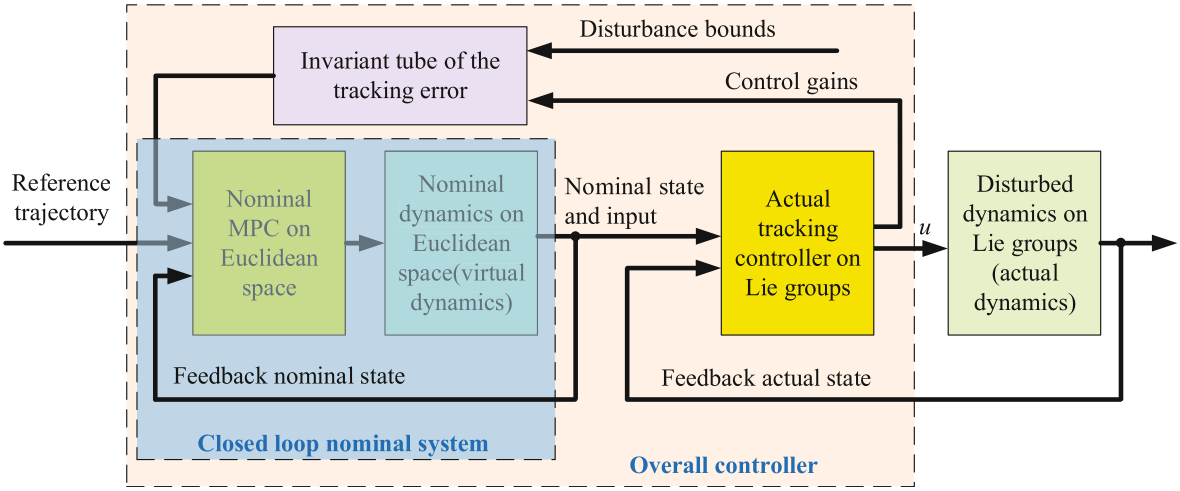

FIGURE 2 The architecture of the overall proposed system

3 | CONTROL SCHEME DESIGN

3.1 | Nominal MPC on ambient euclidean space

By excluding the disturbance from the actual system, the nominal EOM of the system is given by,

$$\dot{\bar{g}} = \bar{g}\bar{\xi}$$
$$\dot{\bar{\xi}} = f(\bar{\xi}, \bar{u}) \tag{13}$$

where the overbar $\bar{*}$ represents the nominal value.

We will solve Problem 1 inspired by the idea of tube-based MPC. The tube-based MPC is composed of a nominal MPC and a feedback controller. The nominal MPC is designed from the nominal tracking error dynamics. By embedding the nominal EOM into Euclidean space, we design the nominal tracking error as $\bar{E} = \bar{x}g_0^{-1} - I \in \mathbb{R}^{n\times n}, \bar{\Xi} = \bar{\xi} - \xi_0$. Then excluding the disturbance from (11), the nominal tracking error dynamics embedded into the Euclidean space is obtained as,

$$\dot{\bar{E}} = (g_0 + \bar{E}g_0)\bar{\Xi}g_0^{-1} - \alpha\nabla V(g_0 + \bar{E}g_0)g_0^{-1}$$
$$\dot{\bar{\Xi}} = f(\bar{\Xi} + \bar{\xi}_0, u) - f(\bar{\xi}_0, \bar{u}_0). \tag{14}$$

Also, we linearize (14) along the reference trajectory and obtain,

$$\dot{\bar{E}}^{\perp} = -\alpha((\nabla^2 V(I)\cdot \bar{E}^{\perp})(g_0 g_0^{\perp})^{-1})^{\perp}$$
$$\dot{\bar{E}}^{\parallel} = g_0\bar{\Xi}g_0^{-1} - \alpha\,(\nabla^2 V(I)\cdot \bar{E}^{\perp})(g_0 g_0^{T})^{-1}\big)^{\parallel}$$
$$\dot{\bar{\Xi}} = \frac{\partial f}{\partial \xi}(\xi_0, u_0)\bar{\Xi} + \frac{\partial f}{\partial u}\bar{\delta}_u. \tag{15}$$

In the nominal MPC design, we define the initial tracking error $\bar{E} = E$ and $\bar{\Xi} = \Xi$, that is, we let $\bar{g} = g$ and $\bar{\xi} = \xi$ at the initial time. The purpose of the nominal MPC is to let $\bar{E}^{\parallel}$ converge to the origin while satisfying the nominal input and state constraints.

To deal with the state and input constraints, we express the admissible set of the configuration and velocity error as $\overline{\mathcal{X}}_e$ and $\overline{\mathcal{V}}_e$, and express the admissible control input set as $\bar{U}_e$. Then the nominal MPC is written as,

$$\begin{aligned}
&\min_{\overline{\delta}u(s)} J(\overline{\zeta},\overline{\delta}_u) = \phi_r(\overline{\zeta}(t_k+\Gamma)) + \int_{t_k}^{t_k+\Gamma} N_r(\overline{\zeta}(s),\overline{\delta}_u(s))ds \\
&s.t.\ \dot{\bar{E}}^{\parallel} = \bar{g}_0\overline{\Xi}\bar{g}_0^{-1} - \alpha\left(\nabla^2 V(I)\cdot \bar{E}^{\perp})(\bar{g}_0\bar{g}_0^T)^{-1}\right)^{\parallel} \\
&\dot{\overline{\Xi}} = \frac{\partial f}{\partial \xi}(\xi_0,u_0)\overline{\Xi} + \frac{\partial f}{\partial u}\overline{\delta}_u \\
&(\bar{E}^{\parallel},\overline{\Xi}) \in \overline{\mathcal{X}}_e \times \overline{\mathcal{V}}_e, \overline{\delta}_u(s) \in \overline{\mathcal{U}}_e, \overline{\zeta}(t_k+\Gamma) \in \Omega_r
\end{aligned} \tag{16}$$

where $\zeta = (E^{\parallel},\Xi)$ is the state, $\phi_r(\cdot)$ and $N_r(\cdot)$ are positive definite functions used to ensure the stability of the MPC, Ω_r is the terminal set will be defined later. Notice that $\overline{\mathcal{X}}_e$, $\overline{\mathcal{V}}_e$, and $\overline{\mathcal{U}}_e$ will be derived later, according to the actual admissible state and input set, as well as the feedback controller.

In order to derive the stability of the system, we introduce the following lemma which is trivial from the Lyapunov second theorem.[36]

Lemma 1. *If $u = u(t,x)$ is an asymptotically tracking controller for the ambient system (15) for the reference trajectory $x_0(t)$, then it is also an asymptotically tracking controller for the system (14) on G for the same reference trajectory.*

3.2 | Feedback control for the disturbed system on matrix Lie groups

The nominal MPC can generate the nominal trajectory of the system on the matrix Lie group. Suppose the nominal error trajectory is given by $\mathbb{R} \ni t \mapsto (\bar{E},\overline{\Xi})$, and the nominal input error trajectory is denoted by $\mathbb{R} \ni t \mapsto \overline{\delta_u}(t) \in \mathbb{R}^m$. Then the nominal state trajectory is obtained as $\bar{g} = (\bar{E}+I)g_0, \bar{\xi} = \xi_0 + \overline{\Xi}$, and the nominal input trajectory is obtained as $\bar{u} = u_0 + \overline{\delta}_u$.

It is noted that using the nominal MPC, the generated nominal state trajectory is already restricted on the matrix Lie group. Therefore, we design the feedback control for the actual systems on the matrix Lie group directly.

For the actual system with uncertainties, it is necessary to design the tracking error carefully. We first define the tracking error between the nominal state and the actual state as $\tilde{E} = g\bar{g}^{-1} - I, \tilde{\Xi} = \xi - \bar{\xi}$. The feedback controller should ensure the boundedness of the tracking error $(\tilde{E},\tilde{\Xi})$ and the input error $\tilde{u} = u - \bar{u}$ so that the constraints in the nominal MPC can be derived from the actual admissible input and state sets.

As the nominal trajectory always evolves on the matrix Lie group, the feedback controller can be designed in a cascaded format. Given the nominal trajectory generated by the nominal MPC, design the velocity ξ_r which is the output of the outer loop controller such that

$$\mathrm{Ad}_{\bar{g}}(\xi_r - \bar{\xi}) = -k_g(\tilde{E}^T\tilde{E} + \tilde{E}^T)^{\parallel} \tag{17}$$

where k_g is a positive constant.

In order to design the control law of the inner loop, we first construct the isomorphic map from $\mathfrak{g}$ to $\mathbb{R}^m$. Suppose the basis of $\mathfrak{g}$ is $\{\sigma_i\}, i = 1,\cdots,m$, the isomorphic map from $\mathfrak{g}$ to $\mathbb{R}^m$ can be defined by the coordinates of $x \in \mathfrak{g}$ under $\{\sigma_i\}$ as,

$$S(x) : \mathfrak{g} \ni x \mapsto x^S \in \mathbb{R}^m \tag{18}$$

where $x^S = (x_1^S, x_2^S, \cdots, x_m^S)^T$, and x_i^S is subject to $x = \Sigma_{i=1}^m x_i^S \sigma_i$.

The inverse of the above isomorphic map $S(\cdot)$ can be defined accordingly. We denote the inverse map as $S^{-1}(x) : \mathbb{R}^m \ni x \mapsto x^I \in \mathfrak{g}$.

Hence, the system dynamics (3) can be written with respect to ξ^S as,

$$\dot{\xi}^S = f^S(\xi,u) + d^S := f_2(\xi^S,u) + d^S. \tag{19}$$

The disturbance term d^S in (19) is related to the disturbance term d in (3) as follows:

$$d^S = S(d).$$

Hence, with $b_1^S := b_1\|S\|$ where b_1 is the assumed bound on $\|d\|$, we have

$$\|d^S\| \leq b_1^S. \tag{20}$$

Then we design the following control law of the inner loop to let ξ track ξ_r,

$$u = u_r - k_\xi(\xi^S - \xi_r^S) \tag{21}$$

where u_r is obtained by inverting $\dot{\xi}_r = f(\xi_r, u_r)$, and k_ξ is a positive constant.

Lemma 2. *[46] Given two vectors $x, y \in \mathbb{R}^n$ their line segment is defined by $Ls(x,y) := \{\xi : \xi = \theta x + (1-\theta)y, 0 < \theta < 1\}$. Consider a vector valued function $h : \mathbb{R}^n \mapsto \mathbb{R}^m$. Assume that h is differentiable on an open set $S \subseteq \mathbb{R}^n$. Let x, y two points of S such that $Ls(x,y) \subseteq S$. Then, there exist constant vectors $c_1, \cdots, c_m \in Ls(x,y)$ such that,*

$$h(x) - h(y) = \left[\sum_{k=1}^{m}\sum_{j=1}^{n} l_m(k) l_n(j)^T \frac{\partial h_k(c_k)}{\partial x_j}\right](x-y)$$

where h_k represents the k-th component of the vector valued function h, x_j represents the j-th component of x, and the vector $l_n(i) \in \mathbb{R}^n$ is defined by

$$l_n(i) = \left[0, \cdots, 0, \underbrace{1}_{\text{i-th element}}, 0, \cdots, 0\right]^T.$$

In order to apply Lemma 2, we define the following function $J(\xi_r^S, u)$ which is a linear map from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^m$,

$$J(\xi_r^S, u) = \sum_{k=1}^{m}\sum_{j=1}^{m} l_m(k) l_m(j)^T \frac{\partial f_{2,k}(\xi_r^S, u)}{\partial u_j}$$

where $f_{2,k}$ represents the k-th component of the vector valued function $f_2(\cdot)$ in (19), u_j represents the j-th component of u.

Assumption 5. There exists a positive constant J_{min} such that

$$\lambda_{min}\left(\frac{J(\xi^S, u) + J^T(\xi^S, u)}{2}\right) \geq J_{min}, \forall \xi \in \mathcal{V}, u \in \mathcal{U} \tag{22}$$

where $\lambda_{min}(A)$ represents the minimum eigen-value of matrix A.

Assumption 6. The function $f_2(\cdot, u)$ is Lipchitz continuous, that is,

$$\|f_2(\xi_1^S, u) - f_2(\xi_2^S, u)\| \leq L_1\|\xi_1^S - \xi_2^S\|, \forall \xi_1 \in \mathcal{V}, \xi_2 \in \mathcal{V}, u \in \mathcal{U} \tag{23}$$

where L_1 is the Lipchitz constant of the function $f_2(\cdot, u)$.

Remark 3. Assumption 5 actually requires $\left(\frac{J(\xi_r^S,u)+J^T(\xi_r^S,u)}{2}\right)$ is positive definite. This is a sufficient condition for the controllability of systems, and is satisfied by many mechanical systems, the work of Reference 47 is such an example. As $J(\xi^S, u)$ can be derived from the dynamics explicitly, given the equation of the dynamics, we can verify if Assumption 5 is satisfied. And Assumption 6 is a common assumption for system dynamics.

We then define an intermediate tracking error $\xi_e := \xi - \xi_r$. From (19) and Lemma 2, there are $c_1, c_2, \ldots, c_m \in Ls(u, u_r)$ such that the velocity tracking error dynamics is subject to,

$$\begin{aligned}\dot{\xi}_e^S &= f_2(\xi^S, u) - f_2(\xi_r^S, u) + f_2(\xi_r^S, u) - f_2(\xi_r^S, u_r) + d^S \\ &= f_2(\xi^S, u) - f_2(\xi_r^S, u) + \left[\sum_{k=1}^{m}\sum_{j=1}^{m} l_m(k) l_m(j)^T \frac{\partial f_{2,k}(\xi_r^S, c_k)}{\partial u_j}\right](u - u_r) + d^S\end{aligned} \tag{24}$$

where the definition of $f_{2,k}$ is given after Lemma 2.

Defining $\varphi_1 = \frac{1}{2}\langle \xi_e^S, \xi_e^S\rangle$, from Assumptions 5 and 6, we arrive at,

$$\begin{aligned}\dot{\varphi}_1 &= \langle \xi_e^S, \dot{\xi}_e^S\rangle \\ &\le L_1\|\xi_e^S\|^2 + \langle \xi_e^S, J(\xi_r^S, c_k)(u - u_r)\rangle + \langle \xi_e^S, d^S\rangle \\ &\le L_1\|\xi_e^S\|^2 - k_\xi \frac{J + J^T}{2}\|\xi_e^S\|^2 + \langle \xi_e^S, d^S\rangle \\ &\le -(k_\xi J_{min} - L_1)\|\xi_e^S\|^2 + \frac{1}{4\rho_g}\|\xi_e^S\|^2 + \rho_g (b_1^S)^2\end{aligned} \tag{25}$$

where ρ_g is a positive constant. Then it is concluded that $\dot{\varphi}_1 < 0$ if $\|\xi_e^S\| > \frac{\rho_g}{k_\xi J_{min} - L_1 - \frac{1}{4\rho_g}} b_1^S$. If we let $\|\xi_e\| = 0$ at the initial instant, and suppose the control gains satisfy $k_\xi J_{min} - L_1 - \frac{1}{4\rho_g} > 0$, then the velocity tracking error ξ_e^S is bounded by

$$\|\xi_e^S\| \le b_v^S := \frac{\rho_g}{k_\xi J_{min} - L_1 - \frac{1}{4\rho_g}} b_1^S \tag{26}$$

which implies

$$\|\xi_e\| \le b_v \tag{27}$$

where $b_v = \frac{\rho_g}{k_\xi J_{min} - L_1 - \frac{1}{4\rho_g}} b_1^S \|S^{-1}\|$ is calculated from b_v^S based on the map $S(\cdot)$ defined in (18).

Then we have the following proposition.

Proposition 1. *Consider system (3). Suppose the nominal state and input trajectory are generated by solving (16), the control law (17) and (21) are used to track the nominal state, the control gains are appropriately selected such that* $k_\xi J_{min} - L_1 - \frac{1}{4\rho_g} > 0$ *and* $2k_g - \frac{1}{2\rho_\xi} > 0$. *Then the tracking error* $\tilde{E}$ *and* ξ_e *converge to the positively invariant set* $\tilde{\Omega}_E := \left\{\tilde{E} : \|(\tilde{E}^T\tilde{E} + \tilde{E}^T)^{\parallel}\| \le \frac{\sqrt{2\rho_\xi} b_v}{\sqrt{2k_g - \frac{1}{2\rho_\xi}}}\right\}$, $\tilde{\Omega}_\xi := \{\xi_e : \|\xi_e\| \le b_v\}$.

Proof. We define the candidate Lyapunov function as,

$$\varphi_2 = \|g\bar{g}^{-1} - I\|^2 = \langle g\bar{g}^{-1} - I, g\bar{g}^{-1} - I\rangle. \tag{28}$$

Then, taking the time derivative of φ_2 yields,

$$\begin{aligned}\dot{\varphi}_2 &= 2\langle \tilde{E}, g(\xi - \bar{\xi})\bar{g}^{-1}\rangle \\ &= 2\langle \tilde{E}, g(\xi - \xi_r)\bar{g}^{-1}\rangle + 2\langle \tilde{E}, g(\xi_r - \bar{\xi})\bar{g}^{-1}\rangle \\ &= 2\langle \tilde{E}, g\bar{g}^{-1}\mathrm{Ad}_{\bar{g}}\xi_e\rangle + 2\langle \tilde{E}, g\bar{g}^{-1}\mathrm{Ad}_{\bar{g}}(\xi_r - \bar{\xi})\rangle \\ &= 2\langle \tilde{E}, (\tilde{E} + I)\mathrm{Ad}_{\bar{g}}\xi_e\rangle + 2\langle \tilde{E}, (\tilde{E} + I)\mathrm{Ad}_{\bar{g}}(\xi_r - \bar{\xi})\rangle \\ &= 2\langle \tilde{E}^T\tilde{E} + \tilde{E}^T, \mathrm{Ad}_{\bar{g}}\xi_e\rangle + 2\langle \tilde{E}^T\tilde{E} + \tilde{E}^T, \mathrm{Ad}_{\bar{g}}(\xi_r - \bar{\xi})\rangle \\ &= 2\langle (\tilde{E}^T\tilde{E} + \tilde{E}^T)^{\parallel}, \mathrm{Ad}_{\bar{g}}\xi_e\rangle + 2\langle (\tilde{E}^T\tilde{E} + \tilde{E}^T)^{\parallel}, \mathrm{Ad}_{\bar{g}}(\xi_r - \bar{\xi})\rangle.\end{aligned} \tag{29}$$

Substituting (17) into (29) and applying Young's inequality we have,

$$\begin{aligned}\dot{\varphi}_2 &\leq -2k_g\|(\tilde{E}^T\tilde{E}+\tilde{E}^T)^{\parallel}\|^2+\frac{1}{2\rho_\xi}\|(\tilde{E}^T\tilde{E}+\tilde{E}^T)^{\parallel}\|^2+2\rho_\xi\|b_v\|^2\\ &\leq -\left(2k_g-\frac{1}{2\rho_\xi}\right)\|(\tilde{E}^T\tilde{E}+\tilde{E}^T)^{\parallel}\|^2+2\rho_\xi b_v^2\end{aligned}$$

where ρ_ξ is a positive constant. It is seen that $\dot{\varphi}_2 \leq 0$ if $\|(\tilde{E}^T\tilde{E}+\tilde{E}^T)^{\parallel}\| \geq \frac{\sqrt{2\rho_\xi}b_v}{\sqrt{2k_g-\frac{1}{2\rho_\xi}}}$. If we also let $\tilde{E}=0$ at the initial instant, it is then concluded that $\tilde{\Omega}_E := \left\{\tilde{E} : \|(\tilde{E}^T\tilde{E}+\tilde{E}^T)^{\parallel}\| \leq \frac{\sqrt{2\rho_\xi}b_v}{\sqrt{2k_g-\frac{1}{2\rho_\xi}}}\right\}$ is a positively invariant set for the closed-loop system under the control law (17) and (21). ■

Remark 4. From the proof procedure we can also conclude that $\tilde{E}$ converges to the set $\left\{\tilde{E} : \|\tilde{E}^T\tilde{E}+\tilde{E}^T\| \leq \frac{\sqrt{2\rho_\xi}b_v}{\sqrt{2k_g-\frac{1}{2\rho_\xi}}}\right\}$. And from the definition of $\tilde{E}$ we can write $\|\tilde{E}^T\tilde{E}+\tilde{E}^T\|$ as,

$$\|\tilde{E}^T\tilde{E}+\tilde{E}^T\| = \|\tilde{E}^T(\tilde{E}+I)\| = \|\tilde{E}^T g\bar{g}^{-1}\| = \sqrt{\operatorname{tr}(\tilde{E}^T g\bar{g}^{-1}(g\bar{g}^{-1})^T\tilde{E})}. \tag{30}$$

As $g\bar{g}^{-1} \in G$, from Assumption 1 the following equation holds,

$$\beta_{gmin}I \preccurlyeq g\bar{g}^{-1}(g\bar{g}^{-1})^T \preccurlyeq \beta_{gmax}. \tag{31}$$

Then we have,

$$\beta_{gmin}\operatorname{tr}(\tilde{E}^T\tilde{E}) \leq \operatorname{tr}(\tilde{E}^T g\bar{g}^{-1}(g\bar{g}^{-1})^T\tilde{E}) \leq \beta_{gmax}\operatorname{tr}(\tilde{E}^T\tilde{E}). \tag{32}$$

Hence $\|\tilde{E}^T\tilde{E}+\tilde{E}^T\| \leq \frac{\sqrt{2\rho_\xi}b_v}{\sqrt{2k_g-\frac{1}{2\rho_\xi}}}$ implies $\|\tilde{E}\| \leq \frac{\sqrt{2\rho_\xi}b_v}{\sqrt{\beta_{gmin}\left(2k_g-\frac{1}{2\rho_\xi}\right)}}$.

3.3 | Constraints revision from tube

The MPC synthesis should consider the revision of the admissible sets of state and control. As we have shown, the feedback control law is designed such that the tracking error and the input fall into the invariant tube, the state and input constraints for the nominal system can be revised accordingly. In this way, the constraints of the actual system are guaranteed in the presence of tracking error induced by the uncertainties.

From the configuration tracking error invariant set $\tilde{\Omega}_E$, the invariant set of $\tilde{g} = g\bar{g}^{-1}$ can be obtained as $\tilde{\Omega}_g = \tilde{\Omega}_E \oplus \{I\}$. Then the admissible set of $\bar{g}$ can be derived as $\overline{\mathcal{X}} = \mathcal{X} \oslash \tilde{\Omega}_g$, and the admissible set of $\bar{E}$ is expressed as $\overline{\mathcal{X}}_e = \overline{\mathcal{X}} \odot g_0^{-1}(t) \ominus \{I\}$, from which we can further derive the admissible set of the nominal parallel tracking error $\overline{\mathcal{X}}_e^{\parallel}$. And combining the results of the previous subsections, the admissible sets used to express the constraints in the nominal MPC can therefore be revised as,

$$\overline{\mathcal{V}}_e = \mathcal{V}_e \ominus (\tilde{\Omega}_\xi \oplus k_g\tilde{\Omega}_E), \overline{\mathcal{U}}_e = \mathcal{U}_e \ominus k_\xi\tilde{\Omega}_\xi \tag{33}$$

where U_e and $\mathcal{V}_e$ are given by

$$\mathcal{U}_e = \mathcal{U} \ominus u_0(t), \mathcal{V}_e = \mathcal{V} \ominus \xi_0(t). \tag{34}$$

Remark 5. As stated after (15), in order to synthesize the tube-based MPC, the initial nominal state is set to $\bar{g} = g$ and $\bar{\xi} = \xi$. In this case, solving the nominal MPC (16) on Euclidean space, the nominal tracking error $\bar{E}$ and $\bar{\xi}$ will converge to origin, while fulfilling the nominal constraints. This means that the nominal state and the nominal input will converge to the reference trajectory while fulfilling the nominal constraints. And as seen from Proposition 1, the error $\tilde{E}$ and $\tilde{\xi}$ is bounded in a small region containing origin. Then applying (33), the actual constraints can be ensured to fulfill. The detailed algorithm to synthesize the tube-based MPC will be presented via an application example in next section.

3.4 | Terminal controller for nominal system

Definition 1. Consider the nominal system (15) on Euclidean space and the finite time optimal control problem (16). The terminal controller $\overline{\delta}_u$ and the terminal set Ω_r are defined such that under the controller $\overline{\delta}_u$,

1. Ω_r is a positively invariant set,
2. $\dot{\phi}_r + N_r(\overline{\zeta}, \overline{\delta}_u) \leq 0$,
3. $\overline{\delta}_u \in \overline{\mathcal{U}}_e$ for all $\overline{\zeta} \in \Omega_r$.

In order to derive the recursive feasibility and stability of the controlled system, we consider the following control law for the nominal system (15),

$$\overline{\delta}_u = \left(\frac{\partial f}{\partial u}\right)^{-1}\left\{[\overline{\Xi}, \overline{\xi}_0] + \overline{Y} - \frac{\partial f}{\partial \xi}(\overline{\xi}_0, u_0)\overline{\Xi}\right\} \tag{35}$$

where $\overline{Y} = g_0^{-1}\Big(- \dot{\overline{W}} + k_p\bar{E}^{\|} + k_d(g_0\overline{\Xi}g_0^{-1} + \overline{W})\Big)g_0$ with $\overline{W} = -\alpha\,(\nabla^2 V(I)\cdot \bar{E}^{\perp})(g_0 g_0^T)^{-1})^{\|}$, k_p and k_d are positive constants such that the matrix

$$\begin{bmatrix} 0 & I \\ k_p I & k_d I \end{bmatrix}$$

is Hurtwitz.

Define $\overline{\Xi}_2 = g_0\overline{\Xi}g_0^{-1} - \alpha\big((\nabla^2 V(I)\cdot \bar{E}^{\perp})(g_0 g_0^T)^{-1}\big)^{\|}$. Then substituting (35) into the nominal EOM (15) yields,

$$\begin{bmatrix} \dot{\bar{E}}^{\|} \\ \dot{\overline{\Xi}}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ k_p I & k_d I \end{bmatrix}\begin{bmatrix} \bar{E}^{\|} \\ \overline{\Xi}_2 \end{bmatrix}. \tag{36}$$

We define $A = \begin{bmatrix} 0 & I \\ k_p I & k_d I \end{bmatrix}$, since the matrix A is Hurwitz, by solving the Riccati equation $A^T P + PA = -I$ we can obtain a positive symmetric matrix P.

From (15) it is seen that $\dot{W}$ can be derived from the state of $E^{\perp}$, $E^{\|}$ and Ξ. From the boundedness of $\nabla^2 V(I)$ and $g_0 g_0^T$, the following inequality holds by appropriately selecting the constant α,

$$\|\overline{W}\| \leq \gamma_1 \|\bar{E}^{\perp}\| \tag{37}$$

where γ_1 is a positive constant.

Similarly the following inequality holds by appropriately selecting the constant α,

$$\|\dot{\overline{W}}\| \leq \gamma_2 \|\bar{E}^{\perp}\| \tag{38}$$

where γ_2 is a positive constant.

Besides, we have the following inequality because of the boundedness of the reference trajectory,

$$[\overline{\Xi}, \xi_0] \leq \gamma_3 \|\overline{\Xi}\| \tag{39}$$

where γ_3 is a positive constant.

From the control law (35), we have,

$$\|\overline{\delta}_u\| \leq \frac{1}{\beta_{amin}}(\gamma_3\|\overline{\Xi}\| + \|\dot{\overline{W}}\| + k_p\|\bar{E}^{\parallel}\| + k_d\|\overline{\Xi}_2\|) + \frac{\beta_{bmin}}{\beta_{amin}}\|\overline{\Xi}\|. \tag{40}$$

From the definition of Ξ_2 it is seen that

$$\|\overline{\Xi}\| \leq \|\overline{\Xi}_2\| + \|\overline{W}\| \leq \|\overline{\Xi}_2\| + \gamma_1\|\bar{E}^{\perp}\|. \tag{41}$$

Combining (40) and (41) we have,

$$\begin{aligned}
\|\overline{\delta}_u\| &\leq \frac{1}{\beta_{amin}}(\gamma_3\|\overline{\Xi}_2\| + \gamma_3\gamma_1\|\bar{E}^{\perp}\| + \|\dot{\overline{W}}\| + k_p\|\bar{E}^{\parallel}\| + k_d\|\overline{\Xi}_2\|) + \frac{\beta_{bmin}}{\beta_{amin}}\|\overline{\Xi}_2\| + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\|\bar{E}^{\perp}\| \\
&\leq \frac{1}{\beta_{amin}}(\gamma_3\|\overline{\Xi}_2\| + \gamma_3\gamma_1\|\bar{E}^{\perp}\| + \gamma_2\|\bar{E}^{\perp}\| + k_p\|\bar{E}^{\parallel}\| + k_d\|\overline{\Xi}_2\|) + \frac{\beta_{bmin}}{\beta_{amin}}\|\overline{\Xi}_2\| + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\|\bar{E}^{\perp}\| \\
&\leq \left(\frac{\gamma_3}{\beta_{amin}} + \frac{\beta_{bmin}}{\beta_{amin}}\right)\|\overline{\Xi}_2\| + \left(\frac{\gamma_3\gamma_1}{\beta_{amin}} + \frac{\gamma_2}{\beta_{amin}} + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\right)\|\bar{E}^{\perp}\| + \frac{k_p}{\beta_{amin}}\|\bar{E}^{\parallel}\|.
\end{aligned} \tag{42}$$

Then we can further obtain,

$$\|\overline{\delta}_u\|^2 \leq c_1\|\overline{\Xi}_2\|^2 + c_2\|\bar{E}^{\perp}\|^2 + c_3\|\bar{E}^{\parallel}\|^2 \tag{43}$$

where

$$c_1 = \left(\frac{\gamma_3}{\beta_{amin}} + \frac{\beta_{bmin}}{\beta_{amin}}\right)^2 + \frac{\left(\frac{\gamma_3}{\beta_{amin}} + \frac{\beta_{bmin}}{\beta_{amin}}\right)\left(\frac{\gamma_3\gamma_1}{\beta_{amin}} + \frac{\gamma_2}{\beta_{amin}} + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\right)}{4\overline{\rho}_1} + \frac{\left(\frac{\gamma_3}{\beta_{amin}} + \frac{\beta_{bmin}}{\beta_{amin}}\right)\frac{k_p}{\beta_{amin}}}{4\overline{\rho}_2}$$

$$c_2 = \left(\frac{\gamma_3\gamma_1}{\beta_{amin}} + \frac{\gamma_2}{\beta_{amin}} + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\right)^2 + \overline{\rho}_1\left(\frac{\gamma_3}{\beta_{amin}} + \frac{\beta_{bmin}}{\beta_{amin}}\right)\left(\frac{\gamma_3\gamma_1}{\beta_{amin}} + \frac{\gamma_2}{\beta_{amin}} + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\right) + \frac{\left(\frac{\gamma_3\gamma_1}{\beta_{amin}} + \frac{\gamma_2}{\beta_{amin}} + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\right)\frac{k_p}{\beta_{amin}}}{4\overline{\rho}_3}$$

$$c_3 = \left(\frac{k_p}{\beta_{amin}}\right)^2 + \overline{\rho}_3\left(\frac{\gamma_3\gamma_1}{\beta_{amin}} + \frac{\gamma_2}{\beta_{amin}} + \frac{\gamma_1\beta_{bmin}}{\beta_{amin}}\right)\frac{k_p}{\beta_{amin}} + \overline{\rho}_2\left(\frac{\gamma_3}{\beta_{amin}} + \frac{\beta_{bmin}}{\beta_{amin}}\right)\frac{k_p}{\beta_{amin}}$$

with positive constants $\overline{\rho}_1$, $\overline{\rho}_2$ and $\overline{\rho}_3$.

Proposition 2. *Consider the nominal tracking error system (15), define a set* $\Omega_r = \{\overline{\zeta} : \phi_r = \langle \nabla^2 V(I) \cdot \bar{E}^{\perp}, \bar{E}^{\perp}\rangle + h_1\langle \bar{E}^{\parallel}_{aug}, P\bar{E}^{\parallel}_{aug}\rangle \leq \epsilon_r\}$, *where* $E^{\parallel}_{aug} = \begin{bmatrix} E^{\parallel} \\ \Xi_2 \end{bmatrix}$, *$h_1$ is a positive constant, and ϵ_r is a positive constant. Define the positive definite function* $N_r = q_1\|\bar{E}^{\perp}\|^2 + q_2\|\bar{E}^{\parallel}\|^2 + q_3\|\overline{\Xi}\|^2 + r_1\|\overline{\Delta\tau}\|^2$, *where the positive constants q_1, q_2, q_3, r_1, h_1 are subject to*

$$\frac{\alpha\lambda_{max}(\nabla^2 V(I))}{\beta_{gmax}} - q_1 - r_1c_2 \geq 0, h_1 - q_2 - r_1c_3 \geq 0, h_1 - q_3 - r_1c_1 \geq 0.$$

If $\overline{\zeta}(t_k + \Gamma) \in \Omega_r$, the control law $\overline{\delta}_u(s), s \in (t_k + \Gamma, t_{k+1} + \Gamma)]$ is designed by (35), where the parameters are appropriately selected, then the following propositions hold for all $s \in (t_k + \Gamma, t_{k+1} + \Gamma)]$,

1. *Ω_r is a positively invariant set,*
2. $\dot{\phi}_r + N_r(\overline{\zeta}, \overline{\delta}_u) \leq 0$,
3. *$\overline{\delta}_u \in \overline{S}$ for all $\overline{\zeta} \in \Omega_r$, where* $\overline{S} = \left\{\overline{\delta}_u : \|\overline{\delta}_u\|^2 \leq \epsilon_r \min\left(\frac{\lambda_{max}(\nabla^2 V(I))}{c_2}, \frac{h_1\lambda_m(P)}{c_1}, \frac{h_1\lambda_m(P)}{c_3}\right)\right\}$.

Proof. From the Definition of ϕ_r, it is trivial to conclude that ϕ_r is positive definite.

Taking the time derivative of ϕ_r yields,

$$
\begin{aligned}
\dot{\phi}_r &= -\alpha\langle \nabla^2 V(I)\cdot E^{\perp}, \left((\nabla^2 V(I)\cdot E^{\perp})(g_0 g_0^T)^{-1}\right)^{\perp}\rangle + h_1\langle \dot{\bar{E}}^{\parallel}_{aug}, P\bar{E}^{\parallel}_{aug}\rangle + h_1\langle \bar{E}^{\parallel}_{aug}, P\dot{\bar{E}}^{\parallel}_{aug}\rangle \\
&= -\alpha\left\langle \nabla^2 V(I)\cdot E^{\perp}, \left((\nabla^2 V(I)\cdot E^{\perp})(g_0 g_0^T)^{-1}\right)^{\perp}\right\rangle + h_1\langle A\bar{E}^{\parallel}_{aug}, P\bar{E}^{\parallel}_{aug}\rangle + h_1\langle \bar{E}^{\parallel}_{aug}, PA\bar{E}^{\parallel}_{aug}\rangle \\
&= -\alpha\left\langle \nabla^2 V(I)\cdot E^{\perp}, \left((\nabla^2 V(I)\cdot E^{\perp})(g_0 g_0^T)^{-1}\right)\right\rangle + h_1\langle \bar{E}^{\parallel}_{aug}, (A^T P + PA)\bar{E}^{\parallel}_{aug}\rangle \\
&\le -\frac{\alpha}{\beta_{gmax}}\left\langle \nabla^2 V(I)\cdot E^{\perp}, \left((\nabla^2 V(I)\cdot E^{\perp})\right)\right\rangle - h_1\left\|\bar{E}^{\parallel}_{aug}\right\|^2 \\
&\le -\frac{\alpha\lambda_{max}(\nabla^2 V(I))}{\beta_{gmax}}\|\bar{E}^{\perp}\|^2 - h_1\left\|\bar{E}^{\parallel}_{aug}\right\|^2.
\end{aligned} \tag{44}
$$

Therefore, it is shown that $\dot{\phi}_r \le 0$, indicating that Ω_r is a positively invariant set.

Secondly, substituting the definition of N_r and (43) into the equation $\dot{\phi}_r + N_r$ we have

$$
\dot{\phi}_r + N_r \le -\left(\frac{\alpha\lambda_{max}(\nabla^2 V(I))}{\beta_{gmax}} - q_1 - r_1 c_2\right)\|\bar{E}^{\perp}\|^2 - (h_1 - q_2 - r_1 c_3)\|\bar{E}^{\parallel}\|^2 - (h_1 - q_3 - r_1 c_1)\|\Xi_2\|^2. \tag{45}
$$

It is seen that $\dot{\phi}_r + N_r \le 0$ if the parameters are selected appropriately.

Thirdly, if $\bar{\zeta} \in \Omega_r$, from (43) and the definition of Ω_r it is obvious that $\bar{\delta}_u \in \bar{S}$.

This completes the proof. ∎

Assumption 7. The reference trajectory (g_0, ξ_0, u_0), the admissible state set $\mathcal{X} \times \mathcal{V}$, the admissible input set $\mathcal{U}$, the parameters in actual controller (21), and the parameters in control law (35) are designed such that $\Omega_r \subset \overline{\mathcal{X}}_e \times \overline{\mathcal{V}}_e$, and $\overline{S} \subset \overline{\mathcal{U}}_e$.

Under Assumption 7 and from Proposition 2, it is seen that the terminal controller for the nominal system (15) can be constructed by the control law (35).

3.5 | Main results

By constructing the terminal controller of the nominal system, it is possible to conclude the feasibility and stability of the nominal system, and then extend the results to the entire controlled system. We have the following theorem which summarizes the recursive feasibility and ISS of the overall closed loop disturbed system.

Theorem 1. *Consider the system dynamics (3). The nominal state $\bar{\zeta}$ and input $\bar{\delta}_u$ is solved from the nominal finite time optimal control problem (16). The feedback control law u is designed by (21). Suppose that at initial time instant the finite time optimal control problem is feasible. Then the closed loop system is input-to-state stable (ISS) with respect to the disturbances.*

Proof. The proof will be finished in two steps: the feasibility proof and the convergence proof.

First, let us prove the feasibility of the MPC problem recursively.

Assume at a sampling instant t_k the solution for the optimal control problem (16) exists, denoted by $\bar{\delta}^*_{u,0}(s), s \in [t_k, t_k + \Gamma]$. According to the state constraints definition, the state $\bar{\zeta}(s)$ stays in the admissible set $\overline{\mathcal{X}}_e \times \overline{\mathcal{V}}_e$ for all $s \in [t_k, t_k + \Gamma]$, and $\bar{\zeta}(t_k + \Gamma) \in \Omega_r$ under the control of $\bar{\delta}^*_{u,0}(t_k)$.

Then for the time interval $[t_{k+1}, t_{k+1} + \Gamma]$ we can construct the following solution for the optimal control problem (16),

$$
\bar{\delta}_{u,0,f} = \begin{cases} \bar{\delta}^*_{u,0}(s), s \in [t_{k+1}, t_k + \Gamma] \\ \bar{\delta}_{u,0,ter}(s), s \in (t_k + \Gamma, t_{k+1} + \Gamma] \end{cases} \tag{46}
$$

where $\bar{\delta}_{u,0,ter}$ is designed by (35).

As $\bar{\delta}_{u,0,ter}$ defined by (35) is a terminal controller, according to Proposition 2, we can conclude that,

$$\bar{\delta}_{u,0,ter}(s) \in \bar{S}, \forall s \in (t_k + \Gamma, t_{k+1} + \Gamma]. \tag{47}$$

Similarly, according to Proposition 2 it is seen that under the control $\bar{\delta}_{u,0,ter}$ we have,

$$\bar{\zeta}(s) \in \Omega_r, \forall s \in (t_k + \Gamma, t_{k+1} + \Gamma]. \tag{48}$$

Combining (47) and Assumption 7 yields,

$$\bar{\delta}_{u,0,ter}(s) \in \overline{\mathcal{U}}_e, \forall s \in (t_k + \Gamma, t_{k+1} + \Gamma]. \tag{49}$$

And Combining (48) and Assumption 7 we can conclude that under the control $\bar{\delta}_{u,0,ter}$,

$$\bar{\zeta}(s) \in \overline{\mathcal{X}}_e \times \overline{\mathcal{V}}_e, \forall s \in (t_k + \Gamma, t_{k+1} + \Gamma], \quad \text{and} \quad \bar{\zeta}(t_{k+1} + \Gamma) \in \Omega_r. \tag{50}$$

From (49) to (50) it is indicated that $\bar{\delta}_{u,0,f}$ is a feasible solution for (16) in the time interval $[t_{k+1}, t_{k+1} + \Gamma]$.

Moreover, as the actual feedback controller is given by (21), by applying Proposition 1, it is concluded that,

$$\zeta(s) \in \mathcal{X}_e \times \mathcal{V}_e, \forall s \in [t_{k+1}, t_{k+1} + \Gamma] \tag{51}$$

and

$$\delta_{u,0,f}(s) \in \mathcal{U}_e, \forall s \in [t_{k+1}, t_{k+1} + \Gamma]. \tag{52}$$

Therefore, we can conclude that $u(s) = u_0(s) + \delta_{u,0,f}(s), s \in [t_{k+1}, t_{k+1} + \Gamma]$ is a feasible control of (3) to satisfy the state and input constraints.

The above derivation shows that the feasibility of a solution of (16) at time t_k implies the feasibility of that at time instant t_{k+1}. The feasibility of the solution of (16) at all time $t > t_k$ can thus be guaranteed recursively in this way.

Next, we consider the convergence of the closed loop overall system.

We define the following Lyapunov candidate of the closed loop *nominal* system,

$$\bar{V}_r = J(\bar{\zeta}, \bar{\delta}_u). \tag{53}$$

Then the difference of $\bar{V}_r$ from the time instant t_k to the time instant t_{k+1} is given by,

$$\begin{aligned}
\Delta \bar{V} &= \bar{V}_r(t_{k+1}) - \bar{V}_r(t_k) \\
&= \int_{t_{k+1}}^{t_{k+1}+\Gamma} \left[N_r\left(\bar{\zeta}(s), \bar{\delta}_u(s)\right)\right] ds - \int_{t_k}^{t_k+\Gamma} \left[N_r\left(\bar{\zeta}(s), \bar{\delta}_u(s)\right)\right] ds \\
&\quad + \phi_r\left(\bar{\zeta}(t_{k+1} + \Gamma)\right) - \phi_r\left(\bar{\zeta}(t_k + \Gamma)\right) \\
&= \int_{t_k+\Gamma}^{t_{k+1}+\Gamma} \left[N_r\left(\bar{\zeta}(s), \bar{\delta}_u(s)\right)\right] ds - \int_{t_k}^{t_{k+1}} \left[N_r\left(\bar{\zeta}(s), \bar{\delta}_u(s)\right)\right] ds \\
&\quad + \phi_r\left(\bar{\zeta}(t_{k+1} + \Gamma)\right) - \phi_r\left(\bar{\zeta}(t_k + \Gamma)\right).
\end{aligned} \tag{54}$$

From Proposition 2, integrating $\dot{\phi}_r + N_r$ over $[t_k + \Gamma, t_{k+1} + \Gamma]$ yields,

$$\int_{t_k+\Gamma}^{t_{k+1}+\Gamma} \left[N_r\left(\bar{\zeta}(s)\right)\right] ds + \phi_r\left(\bar{\zeta}(t_{k+1} + \Gamma)\right) - \phi_r\left(\bar{\zeta}(t_k + \Gamma)\right) \leq 0. \tag{55}$$

Substituting (55) into (54) yields,

$$\Delta \overline{V} \leq 0 \tag{56}$$

Therefore, it is concluded that under the nominal MPC, the linearized nominal system (15) is asymptotically stable at origin. From Lemma 1 it is seen that the nominal system (14) is also asymptotically stable under the nominal MPC.

From the stability of the closed loop nominal system, there exists class $\mathcal{KL}$ functions $f_{kl1}(\cdot)$ and $f_{kl1}(\cdot)$ such that,

$$\|\bar{E}\| \leq f_{kl1}(\|\bar{E}(0)\|, t), \forall t > 0 \tag{57}$$

and

$$\|\overline{\Xi}\| \leq f_{kl2}(\|\overline{\Xi}(0)\|, t), \forall t > 0. \tag{58}$$

Then we consider the *actual* system (3).

First consider the velocity error which satisfies,

$$\Xi = \overline{\Xi} + \tilde{\Xi}. \tag{59}$$

From Proposition 1 the bound of $\tilde{\Xi}$ is obtained as,

$$\|\tilde{\Xi}\| = \|\xi - \bar{\xi}\| = \|\xi_e + \xi_r - \bar{\xi}\| \leq \|\xi_e\| + \|\xi_r - \bar{\xi}\| \leq \frac{k_g\sqrt{2\rho_\xi}b_v}{\sqrt{2k_g - \frac{1}{2\rho_\xi}}} + b_v. \tag{60}$$

It is seen that there exists a class $\mathcal{K}$ function $f_{k1}(\cdot)$ such that,

$$\|\tilde{\Xi}\| \leq f_{k1}(b_1^S), \forall t > 0 \tag{61}$$

which implies that there exists a class $\mathcal{K}$ function $f_{k2}(\cdot)$ such that,

$$\|\tilde{\Xi}\| \leq f_{k2}(b_1), \forall t > 0 \tag{62}$$

where $f_{k2}(b_1) := f_{k1}(b_1\|S\|)$ is trivially a class $\mathcal{K}$ function. Finally, combining the result of (58), (59), and (62), it is seen that $\forall t > 0$,

$$\|\Xi\| \leq \|\overline{\Xi}(t) + \tilde{\Xi}(t)\| \leq \|\overline{\Xi}(t)\| + \|\tilde{\Xi}(t)\| \leq f_{k2}(b_1) + f_{kl2}(\|\overline{\Xi}(0)\|, t). \tag{63}$$

Then we consider the configuration error. Similarly from Proposition 1 and Remark 4 it is seen that there exists a class $\mathcal{K}$ function $f_{k3}(\cdot)$ such that,

$$\|\tilde{E}\| \leq f_{k3}(b_1^S), \forall t > 0. \tag{64}$$

Then we have,

$$\|\tilde{E}\| \leq f_{k4}(b_1), \forall t > 0 \tag{65}$$

where $f_{k4}(b_1) := f_{k3}(b_1\|S\|)$ is also trivially a class $\mathcal{K}$ function.

From the definition of the configuration error we have,

$$E = gg_0^{-1} - I = (\tilde{E} + I)(\bar{E} + I) - I = \tilde{E}\bar{E} + \bar{E} + \tilde{E}. \tag{66}$$

Taking the norm of the both sides of (66) we have,

$$\|E\| \leq \|\tilde{E}\|\|\bar{E}\| + \|\bar{E}\| + \|\tilde{E}\| \leq \rho_E\|\tilde{E}\|^2 + \frac{1}{4\rho_E}\|\bar{E}\|^2 + \|\bar{E}\| + \|\tilde{E}\| \tag{67}$$

where ρ_E is a positive constant.

We define $f_{k5}(\|\tilde{E}\|) := \rho_E\|\tilde{E}\|^2 + \|\tilde{E}\|$ and $f_{k6}(\|\bar{E}\|) := \frac{1}{4\rho_E}\|\bar{E}\|^2 + \|\bar{E}\|$. Both of $f_{k5}(\cdot)$ and $f_{k6}(\cdot)$ are class $\mathcal{K}$ functions, then from (67) it is seen E satisfies,

$$\|E\| \leq f_{k5}(\|\tilde{E}(t)\|) + f_{k6}(\|\bar{E}(t)\|) \leq f_{k7}(b_1) + f_{kl3}(\|\bar{E}(0)\|, t), \forall t > 0. \tag{68}$$

It is seen that $f_{k7} = f_{k5}(f_{k4}(\cdot))$ is a class $\mathcal{K}$ function, and $f_{kl3} = f_{k6}(f_{kl1}(\cdot))$ is a class $\mathcal{KL}$ function.

Finally, from the definition of ζ, as well as (63) and (68), we have the following equation which implies the ISS of the entire system with respect to disturbances,

$$\|\zeta(t)\| \leq f_{k,\zeta}(b_1) + f_{kl,\zeta}(\|\overline{\zeta}(0)\|, t), \forall t > 0 \tag{69}$$

where $f_{k,\zeta}(\cdot)$ is a class $\mathcal{K}$ function, and $f_{kl,\zeta}(\cdot)$ is a class $\mathcal{KL}$ function. ■

Remark 6. Theorem 1 shows the ISS of the system under the sampled nominal MPC and the continuous feedback controller. However, the stability of time-varying sampled data system under the MPC law is still an open problem, and will not be discussed in this paper.

4 | APPLICATION EXAMPLE

In this section, we will take the rotational motion of the rigid body as an application example to illustrate the theoretical results of this paper.

4.1 | Rotational dynamics of rigid body

The attitude control of the rigid body is started from the rotational motion of the rigid body, which is given by,

$$\begin{aligned} \dot{R} &= R\hat{\omega} \\ \dot{\omega} &= M^{-1}(\tau - \hat{\omega}M\omega) + d_r \end{aligned} \tag{70}$$

where $R \in SO(3)$ is the rotation matrix of the rigid body, $\omega \in \mathbb{R}^3$ is the angular velocity, $M \in \mathbb{R}^{3\times3}$ is the inertia tensor, $\tau \in \mathbb{R}^3$ is the torque, and $d_r \in \mathbb{R}^3$ is the disturbance bounded by $\|d_r\| \leq b_r$ with positive constant b_r, and the hat map $\hat{*}$ is an isomorphism from $\mathbb{R}^3$ to the Lie algebra $so(3)$ of Lie group $SO(3)$.[48] For a vector $a = (a_1, a_3, a_3) \in \mathbb{R}^3$, the hat map is given by

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

The inverse map of the hat map is called the vee map and denoted by $\vee$ such that $(\hat{x})^\vee = x$ for all $x \in \mathbb{R}^3$. Considering a function $W \times \mathbb{R}^3 \ni (X, \Omega) \mapsto V(X, \Omega) := \frac{1}{4}\|X^TX - I\|^2 \in \mathbb{R}$, where $W = \{X \in \mathbb{R}^{3\times3} | \det X > 0\}$, then EOM (70) is extended to $\mathbb{R}^{3\times3}$ as,

$$\begin{aligned} \dot{X} &= X\hat{\omega} - \alpha X(X^TX - I) \\ \dot{\omega} &= M^{-1}(\tau - \hat{\omega}M\omega) + d_r \end{aligned} \tag{71}$$

where $(X, \omega) \in \mathbb{R}^{3\times3} \times \mathbb{R}^3$.

Given a reference trajectory,

$$\mathbb{R} \ni t \mapsto (R_0(t), \omega_0(t)) \in SO(3) \times \mathbb{R}^3 \tag{72}$$

and the corresponding reference input torque by

$$\mathbb{R} \ni t \mapsto \tau_0(t) \in \mathbb{R}^3 \tag{73}$$

it is natural to define the error trajectory as,

$$\begin{aligned} &\mathbb{R} \ni t \mapsto (E(t), e(t)) := (X(t)R_0^{-1} - I, \omega(t) - \omega_0(t)) \\ &\in \mathbb{R}^{3\times 3} \times \mathbb{R}^3. \end{aligned} \tag{74}$$

By embedding the manifold into the Euclidean space, and splitting the tracking error E into parallel error $E^{\parallel}$ and transversal error $E^{\perp}$, we have the following linearized tracking error dynamics,

$$\begin{aligned} \dot{E}^{\perp} &= -2\alpha E^{\perp} \\ \dot{E}^{\parallel} &= R_0 \hat{e} R_0^{-1} \\ \dot{e} &= M^{-1}(Me \times \omega_0 + M\omega_0 \times e) + M^{-1}\delta_\tau + d_r \end{aligned} \tag{75}$$

where $\delta_\tau = \tau - \tau_0$. It has been proved that the first error dynamics of (75) is stable, while the second and third components can be stabilized to zeros. Therefore, we can design the MPC from (75).

We suppose that the constraints on the attitude, angular velocity, and input are expressed as,

$$RR_0^T \in \mathcal{X}, \omega \in \mathcal{V}, \tau \in U. \tag{76}$$

Then the admissible sets expressed with respect the error are given by

$$e \in \mathcal{V} \ominus \omega_0(t) := \mathcal{V}_e, \delta_\tau \in U \ominus \tau_0(t) := U_e. \tag{77}$$

4.2 | Feedback control and invariant set of tracking error

For the system constraints (77), we can design the nominal error trajectory using MPC. Suppose the nominal error trajectory is given by $\mathbb{R} \ni t \mapsto (\bar{E}, \bar{e})$, and the nominal input error trajectory is denoted by $\mathbb{R} \ni t \mapsto (\overline{\delta_\tau})$. Then the nominal state and input trajectory is obtained as $\overline{R} = (\bar{E} + I)R_0, \overline{\omega} = \omega_0 + \bar{e}, \overline{\tau} = \tau_0 + \overline{\delta}_\tau$.

As shown in the previous section, we need to design a feedback control law to force the actual trajectory to track the nominal trajectory $(\overline{R}, \overline{\omega}, \overline{\tau})$, and the tracking error should be bounded in a robust invariant set, which is called the tube of the tracking error. A cascaded structure feedback controller will also be considered for this purpose.

First, we design the following reference angular velocity for the feedback attitude control. As the system dynamics always evolves on $SO(3)$, we design the angular velocity ω_r such that,

$$\mathrm{Ad}_{\overline{R}}(\hat{\omega}_r - \hat{\overline{\omega}}) = -k_1 \tilde{E}^{\parallel} \tag{78}$$

where $\tilde{E}^{\parallel} = (R\overline{R}^T - I)^{\parallel} = (R\overline{R}^T - \overline{R}R^T)/2$ is the parallel error between $\overline{R}$ and R, k_1 is a positive constant. Note that $\overline{R}$ always evolves on $SO(3)$ for system (70).

Then design the body torque as,

$$\tau = \tau_r - k_2 e_\omega \tag{79}$$

where $e_\omega = \omega - \omega_r$, $\tau_r = M\dot{\omega}_r + \hat{\omega}_r M \omega_r$, and k_2 is a positive constant.

Let us consider the tracking error of the angular velocity of the rigid body. Substituting (78) into (70) yields,

$$\begin{aligned}\dot{e}_\omega &= M^{-1}\tau - M^{-1}\hat{\omega}M\omega - M^{-1}\tau_r + M^{-1}\hat{\omega}_r M\omega_r + d_r \\ &= M^{-1}(\tau - \tau_r) + M^{-1}\hat{\omega}_r M\omega_r - M^{-1}\hat{\omega}M\omega + d_r.\end{aligned} \tag{80}$$

Define a function $\eta_1(\omega) : \{\omega \in \mathbb{R}^3 : \|\omega\| \le \omega_m\} \ni \omega \mapsto M^{-1}\hat{\omega}M\omega \in \mathbb{R}^3$ with positive constant ω_m, then we have,

$$\|\eta_1(\omega) - \eta_1(\omega_r)\| \le L_2\|e_\omega\| \tag{81}$$

where L_2 is the Lipchitz constant of the function $\eta_1(\cdot)$.

In order to derive the results, we further define $\eta_2(\omega) : \{\omega \in \mathbb{R}^3 : \|\omega\| \le \omega_m\} \ni \omega \mapsto \hat{\omega}M\omega \in \mathbb{R}^3$, hence we have

$$\|\eta_2(\omega_r) - \eta_2(\overline{\omega})\| \le L_3\|\omega_r - \overline{\omega}\| \le L_3 k_1 \|\tilde{E}^{\|}\| \tag{82}$$

where L_3 is the Lipchitz constant of $\eta_2(\cdot)$.

Proposition 3. *Consider the system dynamics (70). The nominal state and input trajectory are represented by $\overline{R}(t), \overline{\omega}(t), \overline{\tau}(t)$. Suppose the control torque is determined by the feedback control law (79). If the positive constants k_1 and k_2 satisfy*

$$\begin{aligned}&k_1 - \frac{1}{4\rho_1} > 0, \\ &k_2\lambda(M)^{-1} - \rho_1 - \frac{1}{4\rho_2} - L_2 > 0\end{aligned}$$

then the state tracking error and the input of the closed-loop system fall into the following sets,

$$\begin{aligned}&\tilde{E}^{\|} \in \tilde{\Omega}_{E^{\|}} = \{\tilde{E}^{\|} : \|\tilde{E}^{\|}\| \le L_R\} \\ &\tilde{\omega} \in \tilde{\Omega}_\omega = \{\tilde{\omega} : \|\tilde{\omega}\| \le (k_1+1)L_R\} \\ &\tilde{\tau} \in \tilde{\Omega}_\tau = \{\tilde{\tau} : \|\tilde{\tau}\| \le (\|M\|k_1(k_1+1) + L_3k_1 + k_2)L_R\}\end{aligned} \tag{83}$$

where $k_5 = \frac{k_3}{k_4}$, $L_R = \sqrt{\frac{\rho_2}{\min(\beta_1,\beta_2)}} b_r$, $\beta_1 = k_1 - \frac{\sqrt{2}}{4\rho_1}$, $\beta_2 = k_2\lambda(M)^{-1} - \rho_1 - \frac{1}{4\rho_2} - L_2$ with positive constants ρ_1, ρ_2, $\lambda(M)$ is the minimum eigenvalue of M.

Proof. We define the following Lyapunov candidate as,

$$\Phi = \frac{1}{2}\|\tilde{E}\|^2 + \frac{1}{2}\|e_\omega\|^2 \tag{84}$$

which is positive definite. From (78), (79) and (80), we can obtain the time derivative of Φ,

$$\begin{aligned}\dot{\Phi} &= \langle \tilde{E}, \dot{\tilde{E}} \rangle + \langle e_\omega, \dot{e}_\omega \rangle \\ &= \langle \tilde{E}, \mathrm{Ad}_{\overline{R}}(\hat{\omega} - \hat{\overline{\omega}}) \rangle + \langle e_\omega, \dot{e}_\omega \rangle \\ &= \langle \tilde{E}, \mathrm{Ad}_{\overline{R}}\hat{e}_\omega \rangle + \langle \tilde{E}, \mathrm{Ad}_{\overline{R}}(\hat{\omega}_r - \hat{\overline{\omega}}) \rangle \\ &\quad - k_2\langle e_\omega, M^{-1}e_\omega \rangle + \langle e_\omega, \eta_1(\omega_r) - \eta_1(\omega) \rangle + \langle e_\omega, d_r \rangle \\ &\le \sqrt{2}\|\tilde{E}^{\|}\|\|e_\omega\| - k_1\|\tilde{E}^{\|}\|^2 - k_2\lambda(M)^{-1}\|e_\omega\|^2 + L_2\|e_\omega\|^2 \\ &\quad + \|e_\omega\|\|d_r\| \\ &\le -\left(k_1 - \frac{\sqrt{2}}{4\rho_1}\right)\|\tilde{E}^{\|}\|^2 \\ &\quad - \left(k_2\lambda(M)^{-1} - \rho_1 - \frac{1}{4\rho_2} - L_2\right)\|e_\omega\|^2 + \rho_2 b_r^2.\end{aligned} \tag{85}$$

Taking $\beta_1 = k_1 - \frac{\sqrt{2}}{4\rho_1}$ and $\beta_2 = k_2\lambda(M)^{-1} - \rho_1 - \frac{1}{4\rho_2} - L_2$, if the parameters are selected such that $\beta_1 > 0$ and $\beta_2 > 0$, then

$$\dot{\Phi} \leq -\min(\beta_1, \beta_2)\|(\tilde{E}^{\|}, e_\omega)\|^2 + \rho_2 b_r^2. \tag{86}$$

It is seen that $\dot{\Phi} < 0$ if $\|(\tilde{E}^{\|}, e_\omega)\| > \sqrt{\frac{\rho_2}{\min(\beta_1,\beta_2)}} b_r$. The bound of $\tilde{E}^{\|}$ and e_ω can be expressed as,

$$\begin{aligned} \|\tilde{E}^{\|}\| &\leq \|(\tilde{E}^{\|}, e_\omega)\| \leq L_R, \\ \|e_\omega\| &\leq \|(\tilde{E}^{\|}, e_\omega)\| \leq L_R. \end{aligned} \tag{87}$$

Recalling the definition of e_ω we arrive at,

$$\|\tilde{\omega}\| \leq k_1\|\tilde{E}^{\|}\| + \|e_\omega\| \leq (k_1 + 1)L_R \tag{88}$$

where $\tilde{\omega} = \omega - \overline{\omega}$.

Then we consider the boundedness of $\tilde{\tau} = \tau_d - \overline{\tau}$. From the control law, we have,

$$\begin{aligned} \|\tau_r - \overline{\tau}\| &= \|M\dot{\omega}_r - M\dot{\overline{\omega}} + \eta_2(\omega_r) - \eta_2(\overline{\omega})\| \\ &\leq \|M\|k_1\|\dot{\tilde{E}}^{\|}\| + L_3 k_1\|\tilde{E}^{\|}\| \\ &\leq \|M\|k_1\|\tilde{\omega}\| + L_3 k_1\|\tilde{E}^{\|}\|. \end{aligned} \tag{89}$$

From (79) it follows that $\tau_d - \tau_r = -k_2 e_\omega$, so combining (87) and (88) we have

$$\begin{aligned} \|\tilde{\tau}\| &= \|\tau_d - \tau_r + \tau_r - \overline{\tau}\| \\ &\leq \|\tau_d - \tau_r\| + \|\tau_r - \overline{\tau}\| \\ &\leq \|M\|k_1\|\tilde{\omega}\| + L_3 k_1\|\tilde{E}^{\|}\| + k_2\|e_\omega\| \\ &\leq (\|M\|k_1(k_1 + 1) + L_3 k_1 + k_2)L_R. \end{aligned} \tag{90}$$

This completes the proof. ■

Remark 7. Proposition 3 gives the robust invariant set of the feedback controlled system in this application example. Similar to Theorem 1, From Proposition 3 we can also prove the ISS of the overall closed loop system w.r.t. the disturbances.

4.3 | Tube-based MPC for rotational motion of rigid bodies

From the invariant set $\tilde{\Omega}_{e_R}$, we can define the invariant set of $\tilde{R} = R\overline{R}^T$ as $\tilde{\Omega}_R = \{\tilde{R} : \|\frac{(\tilde{R}-\tilde{R}^T)^\vee}{2}\| \leq L_R\}$. Because $RR_0^T \in \mathcal{X}$ and $R\overline{R}^T \in \tilde{\Omega}_R$ implies $\overline{R}R_0^T \in \mathcal{X} \oslash \tilde{\Omega}_R$, we can derive the admissible set of $\overline{R}R_0^T$ as $\mathcal{X} \oslash \tilde{\Omega}_R$ in the nominal MPC.

It is noted that $\bar{E}^{\|}$ is not $\overline{R}R_0^T$ itself. As in the nominal MPC, the constraints on the configuration will be expressed with $\bar{E}^{\|}$, we need to further derive the admissible set of $\bar{E}^{\|}$ from the admissible set of $\overline{R}R_0^T$.

Suppose $(\bar{E}^{\|})^\vee = (a_1, a_2, a_3)^T$, the relationship between $\overline{R}R_0^T$ and $\bar{E}^{\|}$ is expressed by,

$$\overline{R}R_0^T = f_R(\bar{E}^{\|}) = \begin{bmatrix} r_{11}(\bar{E}^{\|}) & r_{12}(\bar{E}^{\|}) & r_{13}(\bar{E}^{\|}) \\ r_{21}(\bar{E}^{\|}) & r_{22}(\bar{E}^{\|}) & r_{23}(\bar{E}^{\|}) \\ r_{31}(\bar{E}^{\|}) & r_{32}(\bar{E}^{\|}) & r_{33}(\bar{E}^{\|}) \end{bmatrix} \tag{91}$$

where

$$r_{11} = \frac{(a_2^2 + a_3^2)\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} + a_1^2}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{12} = -a_3 - \frac{a_1\ a_2\ \left(\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} - 1\right)}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{13} = a_2 - \frac{a_1\ a_3\ \left(\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} - 1\right)}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{21} = a_3 - \frac{a_1\ a_2\ \left(\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} - 1\right)}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{22} = \frac{(a_1^2 + a_3^2)\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} + a_2^2}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{23} = -a_1 - \frac{a_2\ a_3\ \left(\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} - 1\right)}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{31} = -a_2 - \frac{a_1\ a_3\ \left(\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} - 1\right)}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{32} = a_1 - \frac{a_2\ a_3\ \left(\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} - 1\right)}{\|(\bar{E}^{\|})^\vee\|^2}$$

$$r_{33} = \frac{(a_1^2 + a_2^2)\sqrt{1 - \|(\bar{E}^{\|})^\vee\|^2} + a_3^2}{\|(\bar{E}^{\|})^\vee\|^2}$$

From (91) the admissible set of $\bar{E}^{\|}$ in the nominal MPC can then be obtained. For example, if the constraint on $\overline{R}$ is expressed by a function $C(\cdot) : SO(3) \ni R \mapsto C(R) \in \mathbb{R}$ as $C(\overline{R}R_0^T) \leq 0$, then we can write the admissible set of $\bar{E}^{\|}$ as $\overline{\mathcal{X}}^{\|} = \{\bar{E}^{\|} : C(f_R(\bar{E}^{\|})) \leq 0\}$, which is used to define the configuration constraint in the nominal MPC.

Combining the previous results, we are now in the position to express the nominal MPC for the rotational motion of the rigid body as,

$$\begin{aligned}
&\min_{\bar{\delta}_\tau(s)} J(\overline{\zeta}, \overline{\delta}_\tau) = \phi_r(\overline{\zeta}(t_k + \Gamma)) + \int_{t_k}^{t_k+\Gamma} \left(N_r(\overline{\zeta}(s), \overline{\delta}_\tau(s))\right) ds \\
&s.t. \quad \dot{\bar{E}}^{\|}(s) = R_0 \hat{\bar{e}} R_0^{-1}, \\
&\dot{\bar{e}} = M^{-1}(M\bar{e} \times \omega_0 + M\omega_0 \times \bar{e}) + M^{-1}\overline{\delta}_\tau \\
&(\bar{E}^{\|}, \bar{e}) \in \overline{\mathcal{X}}^{\|} \times \overline{\mathcal{V}}_e, \overline{\delta}_\tau \in \bar{U}_e
\end{aligned} \tag{92}$$

where $\overline{\zeta} = (\bar{E}^{\|}, \bar{e})$ is the nominal state, $\overline{\mathcal{X}} = \mathcal{X} \oslash \tilde{\Omega}_R$ and $\overline{\mathcal{V}}_e = \mathcal{V}_e \ominus \tilde{\Omega}_\omega$ are the admissible configuration and velocity set, and $\bar{U}_e = U_e \ominus \tilde{\Omega}_\tau$ is the admissible input set.

We then synthesize the tube-based MPC as shown in Algorithm 1. As indicated by Proposition 3, Algorithm 1 combines the nominal MPC which is used to generate the nominal state/input trajectory, and the feedback controller which let the tracking error fall in the tube. The constraints on the state/input for rotational motion of rigid bodies in the presence of uncertainties can therefore be guaranteed to be fulfilled.

Algorithm 1. Synthesis of the proposed tube-based MPC for the constrained attitude control

Initialization: At time instant t_0, let $\bar{\zeta}(0) = \zeta(0)$.

1: At time instant t_k, solve the nominal MPC problem (92), obtain the nominal state and input $\bar{E}^{\parallel}(s), \bar{e}(s), \bar{\delta}_\tau(s), s \in [t_k, t_k + \Gamma)$.
2: Calculate $\bar{R}(s), \bar{\omega}, \bar{\tau}, s \in [t_k, t_k + \Gamma)$.
3: **for** $s \in [t_k, t_{k+1})$ **do**
4: Apply the actual control input $\tau(s)$ to the rigid body, according to (79).
5: **end for**
6: $\left(\zeta(t_k), \bar{\zeta}(t_k)\right) \leftarrow \left(\zeta(t_{k+1}), \bar{\zeta}(t_{k+1})\right)$, $t_k \leftarrow t_{k+1}$.
7: Go to step 1.

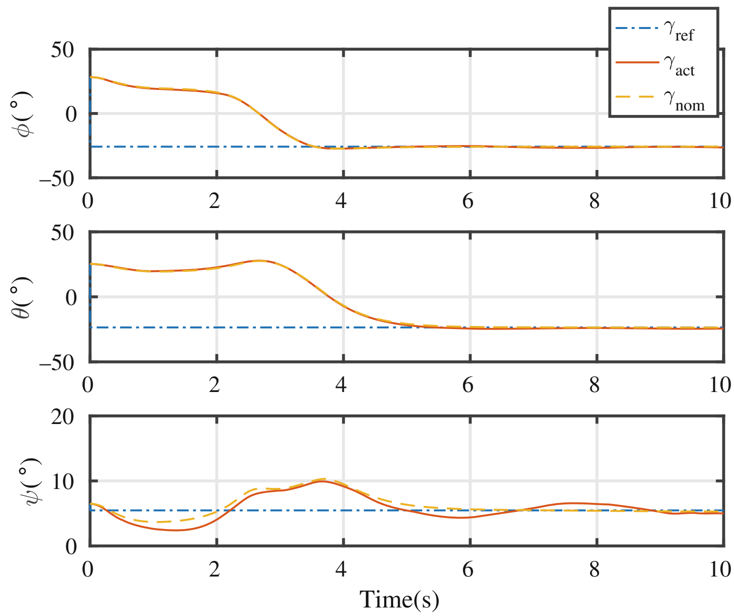

FIGURE 3 The nominal and actual attitude expressed in Euler angles. The dot-dashed line represents the reference value. The solid line represents the actual value. While the virtual line represents the nominal value

4.4 | Simulation

During the simulation, the inertia tensor of the rigid body is $M = \text{diag}(2.263, 2.47, 4.7235) kg \cdot m^2$, the initial attitude of a rigid body is set to $R(0) = \exp(\hat{v}_1)$ with $v_1 = 0.65\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]^T$, and the reference attitude is set to $R_0 = \exp(\hat{v}_2)$ with $v_2 = -0.6\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]^T$. The angular velocity of the rigid body is under the constraint $\|\omega\| < 1$rad/s. While the attitude constraint of the rigid body is given by $0.65 \leq e_3^T R R_0^T R_0 e_3 \leq 0.95$. The disturbance acting on the rigid body is assumed to uniform distribution $d_r \sim U(-1.75, 1.75)$. The Lipchitz constants are calculated according to the EOM as $L_2 = 1.39$ and $L_3 = 3.34$. The open-source ACADO is adopted to solve the MPC problem.[49] In the simulation, the prediction horizon is set to 0.7 s, and the sampling time is 0.1 s.

From Proposition 3, the tube along the nominal attitude trajectory is calculated as $\{e_{\overline{R}} : \|e_{\overline{R}}\| \leq 0.1563\}$, from which the constraint for $\overline{R}R_0^T$ is revised as $0.7608 \leq e_3^T \overline{R} R_0^T R_0 e_3 \leq 0.8895$. And the admissible set for $\bar{E}^{\parallel}$ is further revised as $\{\bar{E}^{\parallel} : 0.7608 \leq e_3^T f_R(\bar{E}^{\parallel}) R_0 e_3 \leq 0.8895\}$ in the nominal MPC.

The simulation results are shown in Figures 3–6. The attitude of the rigid body expressed in Euler angles is shown in Figure 3. It is seen that the attitude of the system evolves from the initial attitude to the reference attitude. The attitude constraint of the rigid body is expressed in Figure 5, from which it is seen that the attitude constraint is satisfied using the

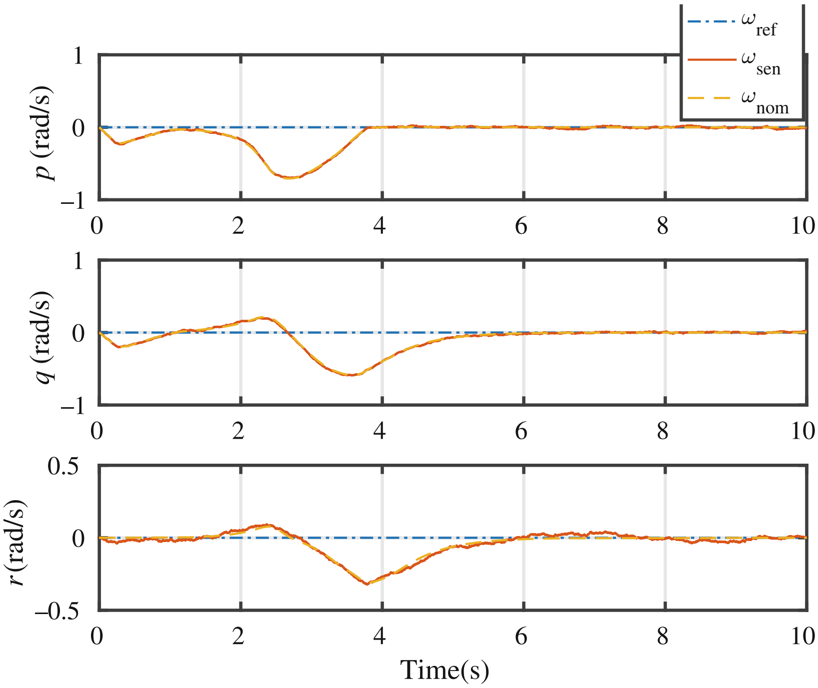

FIGURE 4 The reference and actual angular velocity

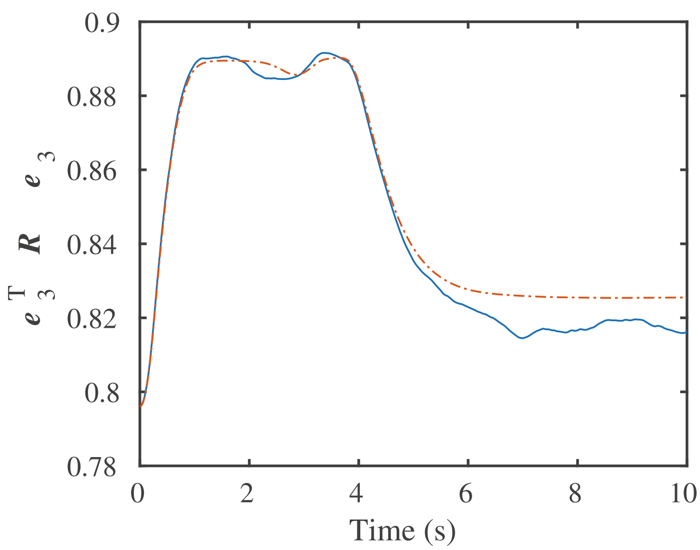

FIGURE 5 The attitude constraints in one test trial

proposed control algorithm, in the presence of uncertainties. It is also noted that because of the attitude constraint, the rotational trajectory from the initial attitude to the desired attitude does not follow the geodesics on $SO(3)$. The angular velocity of the rigid body is depicted in Figure 4. It is seen that the constraints on the angular velocity are also fulfilled. While the input torque under the proposed control algorithm is presented in Figure 6. These two figures also show that the velocity and the input torque all fall in the admissible sets. From the simulation results, the feasibility of the proposed methodology on attitude control of the rigid body is verified.

As a comparison, the base-line normal MPC is tested in the application system also Reference 38. The results under the normal MPC are shown in Figures 7–9. It is seen that the actual attitude and angular velocity can also be

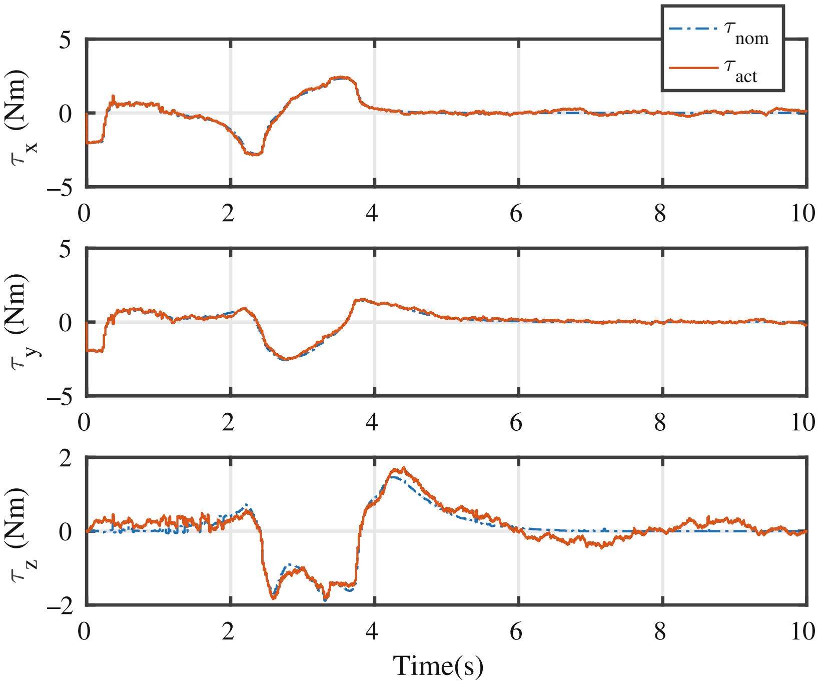

FIGURE 6 The actual and nominal input of the rigid body in one test trial

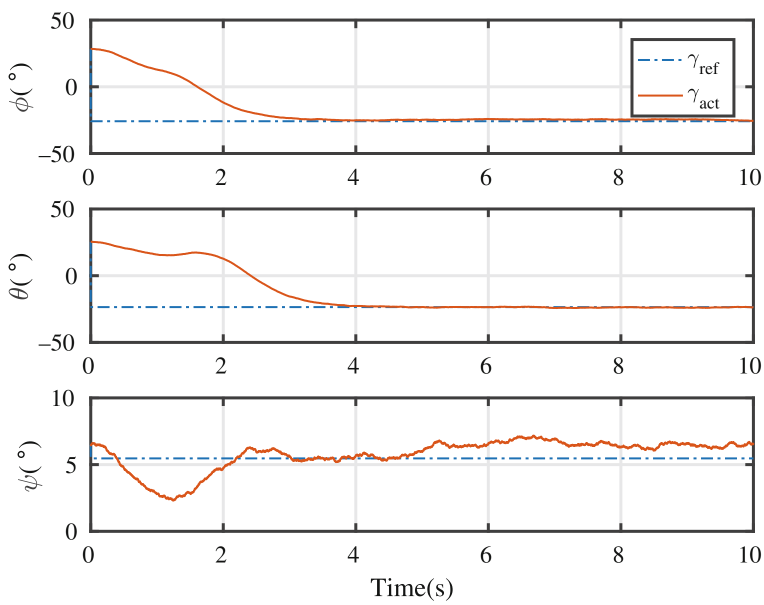

FIGURE 7 The actual attitude evolution in normal MPC

stabilized to the desired attitude and angular velocity under the normal MPC. However, as seen from Figure 9, the attitude constraints are violated due to the unmodeling disturbance in the actual system. As a comparison, our proposed approach is able to fulfill the constraints by tightening the nominal constraints in the nominal MPC. This simulation case demonstrates the advantage of the proposed algorithm over the normal MPC in the presence of the unmodeling disturbances.

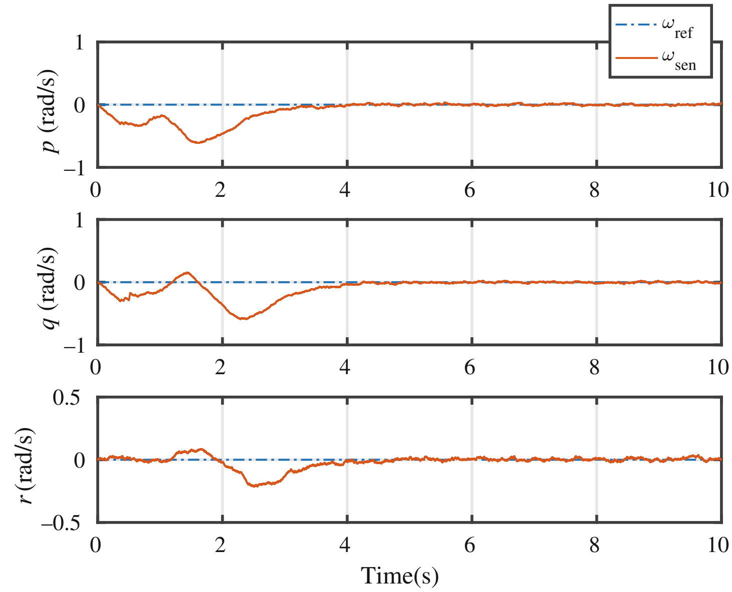

FIGURE 8 The actual angular velocity of the rigid body in one test trial under normal MPC

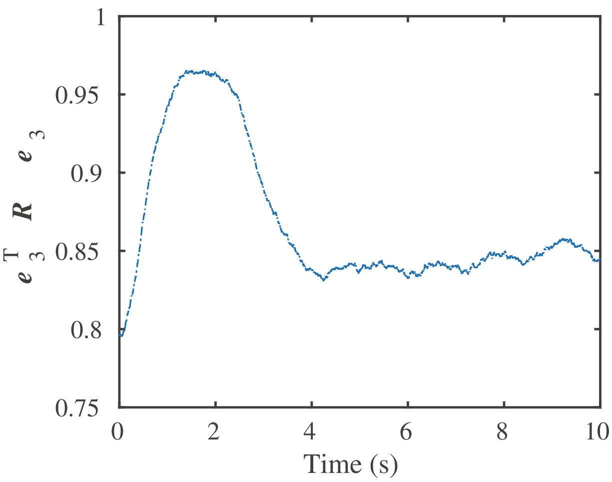

FIGURE 9 The actual attitude constraints of the rigid body in one test trial under normal MPC

5 | CONCLUSIONS

In this paper, we have developed a methodology to design a controller that deals with the state and input constraints for systems on matrix Lie groups with uncertainties. The methodology is inspired by the tube-based MPC. By embedding the manifold into Euclidean space, the nominal MPC has been designed on the Euclidean space. As the generated nominal trajectory is restricted on the Lie group, the feedback controller used to track the nominal trajectory has been designed on the manifold directly. We have shown that the tracking error in the feedback controller can be bounded into robust invariant sets, which can be used to revise the constraints in the nominal MPC expressed in the Euclidean space. In this way, the nominal MPC in the Euclidean space and the feedback controller on the Lie group can be combined together. The proof for the ISS of the overall system evolving on the manifold considering the constraints has been obtained accordingly. It has been proved that the proposed framework can ensure the constraints of systems on Lie groups to be fulfilled in the presence of uncertainties. The application example of the proposed methodology on the rotational motion of the rigid body has been presented. The proposed methodology does not rely on any local coordinates of the Lie group and can apply the existing MPC techniques on the Euclidean space.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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